Coequalizers and Tensor Products for Continuous Idempotent Semirings

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Motivating question

The context-free languages $\mathcal{C}X^*$ over X can be obtained from the regular languages $\mathcal{R}(X \cup Y)^*$ of an extended alphabet $X \cup Y$:

Theorem (Chomsky/Schützenberger 1963)

Consider an alphabet $X \dot\cup Y$, where

- $Y = \{b, d, p, q\}$ *consists of two pairs* b, d *and* p, q *of brackets,*
- $D \subseteq (X \cup Y)^*$ *the Dyck-language of well-bracketed strings*
- $h : (X \cup Y)^* \to X^*$ *the bracket-erasing homomorphism.*

Then $\mathcal{C}X^* = \{\, h(R \cap D) \mid R \in \mathcal{R}(X \cup Y)^* \,\}$.

Question: Is $\mathcal{C}X^*$ a (purely algebraic?) function of $\mathcal{R}X^*$ alone?

Content

We provide some algebraic infrastructure to attack the question.

- Monadic operators $\mathcal{A} : \textsc{Monoid} \to \mathbb{D} := \textsc{Dioid}$
- The subcategory $\mathbb{D}\mathcal{A}$ of $\mathbb{D}$ of $\mathcal{A}$-dioids and $\mathcal{A}$-morphisms
- Congruences and coequalizers
- Tensor product
- Application to the motivating question

In the paper also:

- Coproduct
- Free extension

Monadic operators $\mathcal{A} : \mathbb{M} \to \mathbb{D}$

Let $\mathbb{M}$ be the category of monoids $(M, \cdot, 1)$ and homomorphisms.

Let $\mathbb{D}$ be the category of idempotent semirings (dioids) $(D, +, \cdot, 0, 1)$ with semiring homomorphisms as morphisms.

Each dioid D implicitly has a partial order $\leq$ defined by

$$d \leq d' \iff d + d' = d'.$$

The power-set functor $\mathcal{P} : \mathbb{M} \to \mathbb{D}$ assigns to a monoid M a dioid

$$\begin{aligned} \mathcal{P}M &= (|\mathcal{P}M|, \cup, \cdot, \emptyset, \{1\}), \text{ where} \\ A \cdot B &:= \{\, a \cdot b \mid a \in A, b \in B \,\}, \end{aligned}$$

and to each homomorphism $f : M \to N$ a dioid-homomorphism

$$\mathcal{P}f = \lambda A \, \{\, f(a) \mid a \in A \,\} : \mathcal{P}M \to \mathcal{P}N.$$

A monadic opertor $\mathcal{A}$ (Hopkins 2008) is a subfunctor of the power-set functor $\mathcal{P} : \mathbb{M} \to \mathbb{D}$ that satisfies, for each monoid M,

- A_0 $\mathcal{A}M$ is a set of subsets of M: $\mathcal{A}M \subseteq \mathcal{P}M$,
- A_1 $\mathcal{A}M$ contains each finite subset of M: $\mathcal{F}M \subseteq \mathcal{A}M$,
- A_2 $\mathcal{A}M$ is closed under product (hence a monoid),
- A_3 $\mathcal{A}M$ is closed under union of sets from $\mathcal{A}\mathcal{A}M$ (hence a dioid),
- A_4 $\mathcal{A}$ preserves homomorphisms: if $f : M \to N$ is a homomorphism, so is $\mathcal{A}f := \lambda U \; \{ \, f(u) \mid u \in U \, \} : \mathcal{A}M \to \mathcal{A}N$.

Remark: Each monadic operator $\mathcal{A} : \mathbb{M} \to \mathbb{D}$ is the left adjoint of an adjunction between $\mathbb{M}$ and a subcategory $\mathbb{D}\mathcal{A}$ of $\mathbb{D}$.

Example

The power set operator $\mathcal{P}$ is monadic; $\mathbb{D}\mathcal{P}$ is the category of quantales with unit.

Let $\mathcal{F}$ assign to each monoid M its finite subsets. Then $\mathcal{F}$ is monadic; $\mathbb{D}\mathcal{F}$ is the category $\mathbb{D}$ of dioids.

Example

For infinite cardinal κ, $\mathcal{P}_\kappa M = \{ X \mid X \subseteq M, |X| \leq \kappa \}$ is a monadic operator; $\mathbb{D}\mathcal{P}_{\aleph_0}$ is the category of closed semirings.

For regular cardinal κ, $\mathcal{F}_\kappa M = \{ X \mid X \subseteq M, |X| < \kappa \}$ is monadic; (A_3) corresponds to regularity.

Example

The sets $\mathcal{R}M$, $\mathcal{C}M$ and $\mathcal{T}M$ of all regular, context-free and Turing subsets of a monoid M can be defined via grammars.

For a *grammar* $G = (Q, S, P)$ *of type* $\mathcal{A} \in \{\mathcal{R}, \mathcal{C}, \mathcal{T}\}$ *over* X[1], define $L(G) \subseteq X^*$ and

$$\mathcal{A}X^* := \{\, L(G) \mid G \text{ is a grammar of type } \mathcal{A} \text{ over } X \,\}$$

as usual. With the canonical homomorphism $h : M^* \to M$, put

$$\mathcal{A}M := \{\, h(L) \mid L \in \mathcal{A}M^* \,\}$$

Hopkins 2008: $\mathcal{F} \leq \mathcal{R} \leq \mathcal{C} \leq \mathcal{T} \leq \mathcal{P}$ are monadic operators.

[1] with finite $P \subseteq Q \times (QX \cup X)$, $Q \times (Q \cup X)^*$, $Q^+ \times (Q \cup X)^*$, respectively

The category $\mathbb{D}\mathcal{A} \subseteq \mathbb{D}$ of $\mathcal{A}$-dioids

An $\mathcal{A}$-dioid is a partially ordered monoid $M = (M, \cdot, 1, \leq)$ which is

- $\mathcal{A}$-complete: every $U \in \mathcal{A}M$ has a supremum $\sum U \in M$, and
- $\mathcal{A}$-distributive: for all $U, V \in \mathcal{A}M$, $\sum(UV) = (\sum U)(\sum V)$.

A map $f : M \to N$ between partially ordered monoids is $\mathcal{A}$-continuous, if for all $U \in \mathcal{A}M$ and $n > (\mathcal{A}f)(U)$ there is some $m > U$ with $n \geq f(m)$, where $x > X$ means that x is an upper bound of X.

An $\mathcal{A}$-morphism is an $\mathcal{A}$-continuous monotone homomorphism.

Let $\mathbb{D}\mathcal{A}$ be the category of $\mathcal{A}$-dioids with $\mathcal{A}$-morphisms.

Prop. (i) Every $\mathcal{A}$-dioid $(M, \cdot, 1, \leq)$ is a dioid $(M, +, \cdot, 0, 1)$ via

$$a + b := \sum\{a, b\}, \qquad 0 := \sum \emptyset.$$

(ii) For $f : M \to N$ between $\mathcal{A}$-dioids M, N:

f is $\mathcal{A}$-continuous iff for all $U \in \mathcal{A}M$: $f(\sum U) = \sum(\mathcal{A}f)(U)$

We treat $\mathcal{A}$-dioids in the dioid-signature: $D = (D, +, \cdot, 0, 1)$.

Prop. For monoids M, $\mathcal{A}M$ is the free $\mathcal{A}$-dioid generated by M.

Theorem

Hopkins 2008: $\mathbb{D}\mathcal{R}$ *is the category of* **-continuous Kleene algebras.*

Previous talk: $\mathbb{D}\mathcal{C}$ *is the category of* μ*-continuous Chomsky algebras.*

Unresolved: $\mathbb{D}\mathcal{T}$*, the category of Markov/Turing/Thue algebras*

A monadic operator $\mathcal{A}$ is the left adjoint of an adjunction $(\mathcal{A}, \widehat{\mathcal{A}}, \eta, \epsilon) : \mathbb{M} \to \mathbb{D}\mathcal{A}$ between $\mathbb{M}$ and $\mathbb{D}\mathcal{A}$, where

- $\widehat{\mathcal{A}} : \mathbb{D}\mathcal{A} \to \mathbb{M}$ is the forgetful functor, and if $M \in \mathbb{M}, D \in \mathbb{D}\mathcal{A}$,
- $\eta_M : M \to \mathcal{A}M$ is $m \mapsto \{m\}$, $\epsilon_D : \mathcal{A}D \to D$ is $U \mapsto \sum U$.

This adjunction gives rise to a monad $T_{\mathcal{A}} = (\widehat{\mathcal{A}} \circ \mathcal{A}, \eta, \mu)$ in $\mathbb{M}$, an endofunctor $T = \widehat{\mathcal{A}} \circ \mathcal{A} : \mathbb{M} \to \mathbb{M}$, where

- the unit $\eta : I \to T$ maps $m \in M$ to $\{m\} \in \mathcal{A}M$,
- the product $\mu : TT \to T$ maps $\mathcal{U} \in \mathcal{A}\mathcal{A}M$ to $\bigcup \mathcal{U} \in \mathcal{A}M$.

$\widehat{\mathcal{A}}$ is called a monadic functor in category theory.

The monad $T = T_{\mathcal{A}} : \mathbb{M} \to \mathbb{M}$ of the adjunction $\mathcal{A} : \mathbb{M} \rightleftarrows \mathbb{D}\mathcal{A} : \widehat{\mathcal{A}}$ gives rise to an Eilenberg-Moore category $\mathbb{M}^T$ of T-algebras.

A T-algebra $\langle M, h \rangle$ consists of an object M and a morphism $h : TM \to M$ of $\mathbb{M}$ such that

$$1_M = h \circ \eta_M : M \to M, \text{ and } h \circ (Th) = h \circ \mu_M : TTM \to M.$$

A T-algebra morphism $f : \langle M, h \rangle \to \langle M', h' \rangle$ is a morphism $f : M \to M'$ in $\mathbb{M}$ such that

$$f \circ h = h' \circ (Tf) : TM \to M'.$$

The comparison functor $K : \mathbb{D}\mathcal{A} \to \mathbb{M}^T$ is an isomorphism, where

$$KD = \langle \widehat{D}, \textstyle\sum_D : T\widehat{D} \to \widehat{D} \rangle, \quad K(f : D \to D') = f : \widehat{D} \to \widehat{D'}$$

Coequalizers

For a dioid D and $U, V \subseteq D$, put $U \simeq V :\iff U^{\leq} = V^{\leq}$, where

$$U^{\leq} := \{\, d \in D \mid d \leq u \text{ for some } u \in U \,\},$$

For a dioid-congruence ρ on D, the set D/ρ of congruence classes is a dioid under the operations defined as expected.

An $\mathcal{A}$-congruence on an $\mathcal{A}$-dioid D is a dioid-congruence ρ s.th. for all $U, V \in \mathcal{A}D$, if $U/\rho \simeq V/\rho$, then $(\sum U)/\rho = (\sum V)/\rho$.

For any $E \subseteq D \times D$, there is a least $\mathcal{A}$-congruence on D above E.

Lemma

Let $q : D \to Q$ be an $\mathcal{A}$-morphism between $\mathcal{A}$-dioids D, Q. Then

$$\mathit{ker}(q) := \{\, (a, b) \mid q(a) = q(b),\ a, b \in D \,\}$$

is an $\mathcal{A}$-congruence on D.

Prop. If D is an $\mathcal{A}$-dioid and ρ an $\mathcal{A}$-congruence on D, then D/ρ is an $\mathcal{A}$-dioid and the canonical map $d \mapsto d/\rho$ is an $\mathcal{A}$-morphism.

Proof.

D/ρ is $\mathcal{A}$-complete: Each $U' \in \mathcal{A}(D/\rho)$ is U/ρ for some $U \in \mathcal{A}D$. Since ρ is an $\mathcal{A}$-congruence,

$$\sum U' := (\sum U)/\rho$$

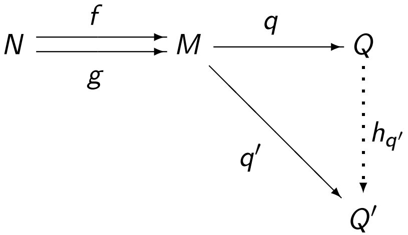
is well-defined, and an upper bound of $U' = U/\rho$.

Let e/ρ be any upper bound of U'. As $\{U, \{e\}\} \in \mathcal{F}\mathcal{A}D \subseteq \mathcal{A}\mathcal{A}D$, we have $U \cup \{e\} \in \mathcal{A}D$. By choice of e, $(U \cup \{e\})/\rho \simeq \{e\}/\rho$, so

$$(e + \sum U)/\rho = (\sum(U \cup \{e\}))/\rho = (\sum\{e\})/\rho = e/\rho.$$

Hence $(\sum U)/\rho \leq e/\rho$, and $\sum U'$ is a least upper bound of U'. □

A coequalizer of two morphisms $f, g : N \to M$ is an object Q with a morphism $q : M \to Q$ such that $q \circ f = q \circ g$ and every morphism $q' : M \to Q'$ with $q' \circ f = q' \circ g$ splits uniquely as shown:



Example

In $\mathbb{M}$, a coequalizer of $f, g : N \to M$ consists of the quotient monoid M/ρ with the canonical map $m \mapsto m/\rho$, where ρ is the least congruence on M above $\{ (f(n), g(n)) \mid n \in N \}$.

Theorem

$\mathbb{D}\mathcal{A}$ has coequalizers. The coequalizer of $f, g : N \to M$ is the canonical map $q : M \to Q$, where $Q = M/\rho$ and ρ is the least $\mathcal{A}$-congruence on M above $E := \{ (f(n), g(n)) \mid n \in N \}$.

Proof.

Q is an $\mathcal{A}$-dioid and q an $\mathcal{A}$-morphism. Clearly, $q \circ f = q \circ g$.

Suppose $q' : M \to Q'$ be an $\mathcal{A}$-morphism with $q' \circ f = q' \circ g$. Define $h : Q \to Q'$ by $h(m/\rho) := q'(m)$, so $q' = h \circ q$.

h is well-defined: since $E \subseteq \ker(q')$, $\rho \subseteq \ker(q')$ by the lemma. h is an $\mathcal{A}$-morphism, because q' is: for $U \in \mathcal{A}M$,

$$h(\sum(U/\rho)) = h((\sum U)/\rho) = q'(\sum U) = \sum\{ h(m/\rho) \mid m \in U \}.$$

As $q : M \to Q = M/\rho$ is surjective, h is unique. □

Prop. If $q : M \to Q$ is a coequalizer of $f, g : N \to M$ in $\mathbb{M}$, then $\mathcal{A}q : \mathcal{A}M \to \mathcal{A}Q$ is a coequalizer of $\mathcal{A}f, \mathcal{A}g : \mathcal{A}N \to \mathcal{A}M$ in $\mathbb{D}\mathcal{A}$.

Theorem

Let E be a congruence on the monoid M, $\mathcal{A}E$ the least $\mathcal{A}$-congruence on $\mathcal{A}M$ above $\{\, (\{m\}, \{m'\}) \mid (m, m') \in E \,\}$. Then $\mathcal{A}M/\mathcal{A}E \simeq \mathcal{A}(M/E)$.

Proof. By the Prop., $\mathcal{A}$ maps the coequalizer in $\mathbb{M}$

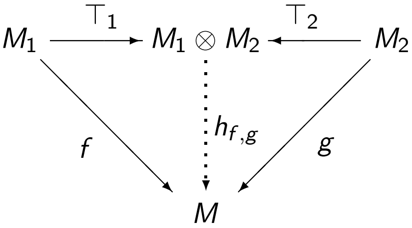
$$M_1 \times M_2 \supseteq \langle E \rangle \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} M \xrightarrow{\cdot/E} M/E$$

to a coequalizer $\mathcal{A}(\cdot/E) : \mathcal{A}M \to \mathcal{A}(M/E)$ of $\mathcal{A}\pi_1, \mathcal{A}\pi_2$ in $\mathbb{D}\mathcal{A}$. Show that $\cdot/\mathcal{A}E : \mathcal{A}M \to \mathcal{A}M/\mathcal{A}E$ is a coequalizer of $\mathcal{A}\pi_1, \mathcal{A}\pi_2$.

Tensor Product

In $\mathbb{M}$, two morphisms $f : M_1 \to M \leftarrow M_2 : g$ are relatively commuting, if for all $m_1 \in M_1, m_2 \in M_2$, $f(m_1)g(m_2) = g(m_2)f(m_1)$.

In a category whose objects have a monoid structure, a tensor product of two objects M_1 and M_2 is an object $M_1 \otimes M_2$ with two relatively commuting morphisms $\top_1 : M_1 \to M_1 \otimes M_2 \leftarrow M_2 : \top_2$ such that for any pair $f : M_1 \to M \leftarrow M_2 : g$ of relatively commuting morphisms the diagram



can be uniquely completed as shown.

Intuitively, $M_1 \otimes M_2$ is a free extension of both objects in which elements of one commute with elements of the other.

Example

In $\mathbb{M}$, the tensor product $\top_1 : M_1 \to M_1 \otimes M_2 \leftarrow M_2 : \top_2$ consists of $M_1 \otimes M_2 := M_1 \times M_2$ with $T_1(m) = (m, 1)$, $T_2(m') = (1, m')$.

Theorem

In the category $\mathbb{D}\mathcal{A}$, the tensor product of $\mathcal{A}$-dioids D_1, D_2 is

$$\top_1 : D_1 \to D_1 \otimes_{\mathcal{A}} D_2 \leftarrow D_2 : \top_2$$

with $D_1 \otimes_{\mathcal{A}} D_2 := \mathcal{A}(M_1 \otimes M_2)/_{\equiv}$ and $\top_k = \pi \circ \eta \circ \hat{\top}_k$, where

- $\equiv$ *is the least $\mathcal{A}$-congruence on $\mathcal{A}(M_1 \otimes M_2)$ above*

$$E := \{ (\{(\sum A, \sum B)\}, A \times B) \mid A \in \mathcal{A}M_1, B \in \mathcal{A}M_2 \},$$

- $\hat{\top}_1 : M_1 \to M_1 \otimes M_2 \leftarrow M_2 : \hat{\top}_2$ *in $\mathbb{M}$ of $M_k = \widehat{\mathcal{A}}D_k$*
- $\eta : M_1 \otimes M_2 \to \mathcal{A}(M_1 \otimes M_2)$ *is $(m_1, m_2) \mapsto \{(m_1, m_2)\}$,*
- π *is the canonical map $U \mapsto U/_{\equiv}$.*

The induced map of $f : D_1 \to D \leftarrow D_2 : g$ is

$$h_{f,g}(U/_{\equiv}) := \sum \{ f(a)g(b) \mid (a, b) \in U \}, \quad U \in \mathcal{A}(M_1 \times M_2).$$

Proof.

Let $f : D_1 \to D \leftarrow D_2 : g$ be relatively commuting $\mathcal{A}$-morphisms. The homomorphism $\hat{h}_{f,g} : M_1 \times M_2 \to D$ with $f = \hat{h}_{f,g} \circ \hat{\top}_1$ and $g = \hat{h}_{f,g} \circ \hat{\top}_2$, i.e. $\hat{h}_{f,g}(a, b) = f(a)g(b)$, extends uniquely to an $\mathcal{A}$-morphism $\hat{h}^*_{f,g} : \mathcal{A}(M_1 \times M_2) \to D$ with $\hat{h}_{f,g} = \hat{h}^*_{f,g} \circ \eta$, by

$$\hat{h}^*_{f,g}(U) = \sum \{ \hat{h}_{f,g}(a, b) \mid (a, b) \in U \}.$$

We have $E \subseteq \ker(\hat{h}^*_{f,g})$, hence $\equiv \subseteq \ker(\hat{h}^*_{f,g})$, so

$$h_{f,g}(U/_{\equiv}) := \hat{h}^*_{f,g}(U), \qquad \text{for } U \in \mathcal{A}(M_1 \times M_2),$$

well-defines $h_{f,g} : D_1 \otimes_{\mathcal{A}} D_2 \to D$.

The needed properties of $h_{f,g}, \top_k$ follow from those of $\hat{h}_{f,g}, \hat{\top}_k$. □

Prop. $\mathcal{A}M_1 \otimes_{\mathcal{A}} \mathcal{A}M_2 \simeq \mathcal{A}(M_1 \otimes M_2)$ for monoids M_1, M_2.

Proof (Sketch) Recall $M_1 \otimes M_2 = M_1 \times M_2$ and use

$$\begin{aligned} U/_{\equiv} &\mapsto \{ (\sum A, \sum B) \mid (A, B) \in U \}, \quad U \in \mathcal{A}(\mathcal{A}M_1 \times \mathcal{A}M_2) \\ V &\mapsto \{ (\{a\}, \{b\}) \mid (a, b) \in V \}/_{\equiv}, \quad V \in \mathcal{A}(M_1 \times M_2). \end{aligned}$$

It is known that $D^{n\times n} \in \mathbb{D}\mathcal{A}$ for $\mathcal{A} \in \{\mathcal{F}, \mathcal{R}, \mathcal{C}, \mathcal{P}\}$.

Prop. If $D^{n\times n} \in \mathbb{D}\mathcal{A}$, then $D^{n\times n} \simeq D \otimes_{\mathcal{A}} \mathbb{B}^{n\times n}$.

Similarly for the coproduct:

Theorem

The coproduct of $D_1, D_2 \in \mathbb{D}\mathcal{A}$ is $D_1 \oplus_{\mathcal{A}} D_2 := \mathcal{A}(M_1 \oplus M_2)/_{\equiv}$.

Prop. $\mathcal{A}M_1 \oplus_{\mathcal{A}} \mathcal{A}M_2 \simeq \mathcal{A}(M_1 \oplus M_2)$ for monoids M_1, M_2.

Return to the question: Is $\mathcal{C}X^*$ a function of $\mathcal{R}X^*$ alone?

The ρ be the $\mathcal{R}$-congruence on $\mathcal{R}Y^*$ generated by the equations[2]

$$bd = 1, \quad pq = 1, \quad bq = 0, \quad pd = 0$$

Then $C = \mathcal{R}Y^*/\rho$ is an $\mathcal{R}$-dioid, the polycyclic $\mathcal{R}$-dioid.

In C, every $w \in Y^* \setminus \{d, q\}^*\{b, p\}^*$ is equivalent to 0 or 1.

Identify $w \in X^*, z \in Y^*$ with their images $\top_1(\{w\}), \top_2(\{z\}/_\rho)$ in the tensor product $\top_1 : \mathcal{R}X^* \to \mathcal{R}X^* \otimes_\mathcal{R} C \leftarrow C : \top_2$.

Theorem

$\mathcal{C}X^* \subseteq \mathcal{R}X^* \otimes_\mathcal{R} C$, in the sense that for each $L \in \mathcal{C}X^*$,

$$\sum L = \sum \{ \top_1(\{w\}) \mid w \in L \} \in \mathcal{R}X^* \otimes_\mathcal{R} C.$$

[2] as usual, $w \in Y^*$ stands for $\{w\} \in \mathcal{R}Y^*$, 0 for $\emptyset$

Corollary

Every context-free language $L \subseteq X^*$ is denoted by a regular expression over $X \cup Y$ in $K := \mathcal{R}X^* \otimes_{\mathcal{R}} C$.

Example

For $L = \{\, u^n v^n \mid n \in \mathbb{N} \,\} \in \mathcal{C}X^*$, $\sum L = b(up)^*(qv)^*d \in K$:

Since K is *-continuous and $x \in X$ commutes with $y \in Y$ in K,

$$\begin{aligned} K \ni \; b(up)^*(qv)^*d &= b(\sum_n (up)^n)(\sum_m (qv)^m)d \\ &= \sum_{n,m} u^n v^m \, bp^n q^m d \\ &= \sum_n u^n v^n = \sum L. \end{aligned}$$

This has implications for recognition/parsing/transduction!

Adding $db + qp \leq 1$ to ρ in $C = \mathcal{R}Y^*/\rho$, one can show:

Theorem (Hopkins, in preparation)

- *For any monoid M,*

$$\mathcal{C}M = \{\, r \in \mathcal{R}M \otimes_{\mathcal{R}} C \mid r \text{ commutes with each } c \in C \,\}.$$

- *There is an adjunction $Q : \mathbb{D}\mathcal{R} \rightleftarrows \mathbb{D}\mathcal{C} : \widehat{Q}$ such that*

$$Q\,D = \{\, r \in D \otimes_{\mathcal{R}} C \mid r \text{ commutes with each } c \in C \,\}$$

is the "fixed-point closure" of the *-continuous Kleene algebra.

Conclusion

We have provided some universal-algebraic infrastructure for classes of idempotent semirings D with subset families $\mathcal{A}D$ where

- $\leq$ is $\mathcal{A}$-complete: each $A \in \mathcal{A}D$ has a supremum $\sum A \in D$,
- $\cdot$ is $\mathcal{A}$-distributive: $\sum(AB) = (\sum A)(\sum B)$ for all $A, B \in \mathcal{A}D$.

Open questions

1. Do the results follow from known results in category theory?
2. Axiomatizations for $\mathbb{D}\mathcal{S}$ and $\mathbb{D}\mathcal{T}$? $\mathcal{T}X^* \subseteq \mathcal{R}X^* \otimes_{\mathcal{R}} C \otimes_{\mathcal{R}} C$?
3. Our $\mathcal{A} : \mathbb{M} \to \mathbb{D}$ use A_1 : "$\mathcal{F} \leq \mathcal{A}$" and involve semirings. A T-algebra construction as for $\mathbb{D}\mathcal{A} \simeq \mathbb{M}^T$ only needs A_1^-: "$\mathcal{I} \leq \mathcal{A}$" for $\mathcal{I}M := \{ \{m\} \mid m \in M \}$ and also works if $\mathbb{D}$ is replaced by $\mathbb{M}^{\leq}$, the partially ordered monoids. Which of the results hold for the subcategories $\mathbb{M}^{\leq}\mathcal{A}$ of $\mathbb{M}^{\leq}$?
4. For $\mathcal{A}$-dioids D, is $D^{n \times n}$ an $\mathcal{A}$-dioid?