C-Dioids and μ-Continuous Chomsky Algebras

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We consider "formal languages" over non-free monoids M.

- transductions $T \subseteq X^* \times Y^*$ ought to be handled in the same formalism as languages, but the monoid $X^* \times Y^*$ is not free: $(x, \epsilon)(\epsilon, y) = (x, y) = (\epsilon, y)(x, \epsilon)$.
- natural languages use non-free concatenation, e.g.
  - sound laws in word formation: $bet + ing = betting$
  - inflection in phrase formation: $few + man = few\ men$, $this\ woman + (to)\ read\ a\ book = this\ woman\ reads\ a\ book$.

We here look at the set of context-free subsets $\mathcal{C}M \subseteq \mathcal{P}M$ of M.

Content

1. C-dioids: idempotent semirings $(M, +, 0, \cdot, 1)$ where
   - sups $\sum U \in M$ of context-free subsets $U \subseteq M$ exist,
   - distributivity: $(\sum U)(\sum V) = \sum(UV)$ for cf-sets U, V.
2. Chomsky-algebras: idempotent semirings $(M, +, 0, \cdot, 1)$ with:
   - CFGs $\bar{x} \geq \bar{p}(\bar{x})$ have least solutions $\mu\bar{x}\bar{p}^M$,
   - μ-continuity: $a \cdot \mu xt^M \cdot b = \sum\{a \cdot mxt^M \cdot b \mid m \in \mathbb{N}\}$

We show: C-dioid $=$ μ-continuous Chomsky algebra.

Main references:

- M. Hopkins. The Algebraic Approach I+II: The Algebraization of the Chomsky Hierarchy + Dioids, Quantales and Monads. In: RelMiCS/AKA, LNCS 4988, pp. 155–190 (2008).
- N. B. B. Grathwohl, F. Henglein, D. Kozen. Infinitary axiomatization of the equational theory of context-free languages. In: Proc. FICS 2013, pp. 44–55 (2013).

0. Definitions

A semiring $R = (R, +, 0, \cdot, 1)$ is a set R with two operations $+, \cdot : R \times R \to R$, such that $(R, +, 0)$ and $(R, \cdot, 1)$ are monoids, $+$ is commutative, and the zero and distributivity laws holds:

$$a0b = 0, \qquad a(b + c)d = abd + acd$$

A dioid or idempotent semiring $D = (D, +, 0, \cdot, 1)$ is a semiring in which $+$ is idempotent. It has a natural partial order $\leq$, defined by

$$a \leq b :\iff a + b = b.$$

A partially ordered monoid $(M, \cdot, 1, \leq)$ is a monoid $(M, \cdot, 1)$ with a partial order $\leq$ and where $\cdot$ is monotone in each argument.

Monadic operators $\mathcal{A}$

If $M = (M, \cdot^M, 1^M)$ is a monoid, its power set $(\mathcal{P}(M), \cdot, 1, \subseteq)$ is a partially ordered monoid –and $(\mathcal{P}(M), \cup, \emptyset, \cdot, 1)$ a dioid–, where

$$A \cdot B := \{a \cdot^M b \mid a \in A, b \in B\}, \quad 1 := \{1^M\}.$$

Let $\mathbb{M}$ and $\mathbb{D}$ be the categories of monoids and dioids, with corresp. homomorphisms. A functor $\mathcal{A} : \mathbb{M} \to \mathbb{D}$ is a monadic operator (Hopkins 2008), if for each monoid M

- A_0 $\mathcal{A}M$ is a set of subsets of M: $\mathcal{A}M \subseteq \mathcal{P}M$,
- A_1 $\mathcal{A}M$ contains each finite subset of M: $\mathcal{F}M \subseteq \mathcal{A}M$,
- A_2 $\mathcal{A}M$ is closed under product (hence a monoid),
- A_3 $\mathcal{A}M$ is closed under union of sets from $\mathcal{A}\mathcal{A}M$ (hence a dioid),
- A_4 $\mathcal{A}$ preserves monoid-homomorphisms: if $f : M \to N$ is a homomorphism, so is $\mathcal{A}f : \mathcal{A}M \to \mathcal{A}N$, where for $U \subseteq M$

$$(\mathcal{A}f)(U) := \{f(u) \mid u \in U\} =: \widetilde{f}(U).$$

Example (Hopkins 2008, algebraic Chomsky-hierarchy)

The operators $\mathcal{F} \leq \mathcal{R} \leq \mathcal{C} \leq \mathcal{T} \leq \mathcal{P}$ are monadic (A_3!):

1. $\mathcal{P}M$ = all subsets of M,
2. $\mathcal{F}M$ = all finite subsets of M,
3. $\mathcal{R}M$ = the closure of $\mathcal{F}M$ under $+$ (union), $\cdot$ (elementwise product) and * (iteration), i.e. $A^* = \bigcup\{A^n \mid n \in \mathbb{N}\}$.
4. $\mathcal{C}M$ = the closure of $\mathcal{F}M$ under least solutions in $\mathcal{P}M$ of systems $x_1 \geq p_1(\bar{x}), \ldots x_n \geq p_n(\bar{x})$ with polynomials $p_i(\bar{x})$ over $\mathcal{C}M$.
5. $\mathcal{T}M$ = all Turing/Thue-subsets $\mathcal{T}M$ of M.

Rem. $\mathcal{S}M$ = all context-sensitive subsets of M is *not* monadic.(A_4)

Remark

A monadic operator $\mathcal{A}$ is the left adjoint of an adjunction $(\mathcal{A}, \widehat{\mathcal{A}}, \eta, \epsilon) : \mathbb{M} \to \mathbb{D}\mathcal{A}$ between $\mathbb{M}$ and a subcategory $\mathbb{D}\mathcal{A}$ of $\mathbb{D}$, where

- $\widehat{\mathcal{A}} : \mathbb{D}\mathcal{A} \to \mathbb{M}$ is the forgetful functor; for $M \in \mathbb{M}$, $D \in \mathbb{D}\mathcal{A}$
- $\eta_M : M \to \mathcal{A}M$ is $m \mapsto \{m\}$, $\epsilon_D : \mathcal{A}D \to D$ is $U \mapsto \sum U$.

This adjunction gives rise to a monad $(\widehat{\mathcal{A}} \circ \mathcal{A}, \eta, \mu)$ in $\mathbb{M}$, an endofunctor $T = \widehat{\mathcal{A}} \circ \mathcal{A} : \mathbb{M} \to \mathbb{M}$, where

- the unit $\eta : I \to T$ maps $m \in M$ to $\{m\} \in \mathcal{A}M$ and
- the product $\mu : TT \to T$ maps $\mathcal{U} \in \mathcal{A}\mathcal{A}M$ to $\bigcup \mathcal{U} \in \mathcal{A}M$.

$\widehat{\mathcal{A}}$ is called a monadic functor in category theory.

$\mathbb{D}\mathcal{A}$ is isomorphic to the Eilenbg-Moore category $\mathbb{M}^T$ of T-algebras

$$(D, +, 0) \mapsto \langle D, \textstyle\sum_D : TD \to D \rangle, \qquad D = (D, \cdot, 1).$$

$\mathcal{A}$-dioids and the subcategory $\mathbb{D}\mathcal{A}$ of $\mathbb{D}$

An $\mathcal{A}$-dioid is a partially ordered monoid $M = (M, \cdot, 1, \leq)$ which is

- $\mathcal{A}$-complete: every $U \in \mathcal{A}M$ has a supremum $\sum U \in M$, and
- $\mathcal{A}$-distributive: for all $U, V \in \mathcal{A}M$, $\sum(UV) = (\sum U)(\sum V)$.

Every $\mathcal{A}$-dioid $(M, \cdot, 1, \leq)$ is a dioid, using $a + b := \sum\{a, b\}$ and $0 := \sum \emptyset$. For monoid M, $\mathcal{A}M$ is the free $\mathcal{A}$-dioid generated by M.

A map $f : M \to N$ between partially ordered monoids is $\mathcal{A}$-continuous, if for all $U \in \mathcal{A}M$ and $n > (\mathcal{A}f)(U)$ there is some $m > U$ with $n \geq f(m)$, where $x > X$ means that x is an upper bound of X.

An $\mathcal{A}$-morphism is an $\mathcal{A}$-continuous monotone homomorphism.

Let $\mathbb{D}\mathcal{A}$ be the category of $\mathcal{A}$-dioids with $\mathcal{A}$-morphisms.

Prop. Let M and N be $\mathcal{A}$-dioids.

(i) A monotone homomorphism $f : M \to N$ is $\mathcal{A}$-continuous iff

$$f(\sum U) = \sum \widetilde{f}(U) \qquad \text{for all } U \in \mathcal{A}M.$$

(ii) Every $\mathcal{A}$-morphism $f : M \to N$ is a dioid-homomorphism.

$\mathcal{A}$-distribuitivity amounts to $\sum$-continuity of $\cdot$ in both arguments. It can be expressed as $\sum$-continuity in each argument separately:

Prop. (Hopkins 2008) If the partially ordered monoid M is $\mathcal{A}$-complete, the following conditions are equivalent:

(i) for all $U, V \in \mathcal{A}M$, $\sum(UV) = \sum U \cdot \sum V$.

(ii) for all $a, b \in M$ and $U \in \mathcal{A}M$, $\sum aUb = a(\sum U)b$.

$\mathcal{R}$-dioids

Theorem (Hopkins 2008)

$\mathbb{DR}$ equals Kozen's category of *-continuous Kleene algebras.

A Kleene algebra $(K, +, 0, \cdot, 1, ^*)$ is an idempotent semiring (dioid) $(K, +, 0, \cdot, 1)$ with an operation $^* : K \to K$ so that for all $a, b \in K$

- $a^* b$ is the least solution of $x \geq ax + b$,
- ba^* is the least solution of $x \geq xa + b$.

The Kleene algebra K is *-continuous, if for all $a, b, c \in K$,

$$ac^* b = \sum \{ac^n b \mid n \in \mathbb{N}\}.$$

In particular:

- K is *-complete: each $U_c = \{c^n \mid n \in \mathbb{N}\}$ has a sup: c^*.
- K is *-distributive: for all a, b, c, $a(\sum U_c)b = \sum(aU_c b)$.

$\mathcal{C}$-dioids

For the free monoid Y^* generated by Y,

- $\mathcal{R}Y^* =$ the semiring of regular languages over Y,
- $\mathcal{C}Y^* =$ the semiring of context-free languages over Y.

We are interested in the category $\mathbb{D}\mathcal{C}$ of $\mathcal{C}$-dioids as a generalization of the theory of context-free languages over free monoids.

Claim. $\mathbb{D}\mathcal{C}$ equals the category of μ-continuous Chomsky-algebras.

For a monoid M, $\mathcal{C}M$ is the closure of $\mathcal{F}M$ under (components of) least solutions (in the complete dioid $\mathcal{P}M$) of polynomial systems

$$x_1 \geq p_1^{\mathcal{P}M}(\bar{x}, \bar{A}), \ldots, x_n \geq p_n^{\mathcal{P}M}(\bar{x}, \bar{A}) \quad \text{with } \bar{A} \in \mathcal{C}M.$$

So $\mathcal{C}$-dioids ought to be closed under a least-fixed-point operator μ.

Partially ordered μ-semirings

Let X be an infinite set of variables and consider μ-terms over X:

$$s, t := x \mid 0 \mid 1 \mid (s \cdot t) \mid (s + t) \mid \mu x\, t$$

A partially ordered μ-semiring $(M, +, \cdot, 0, 1, \leq)$ is a semiring $(M, +, \cdot, 0, 1)$ with a partial order $\leq$ on M, where every term t defines a function $t^M : (X \to M) \to M$, so that

for all terms s, t, variables $x \in X$ and valuations $g, h : X \to M$

1. $$\begin{array}{lllllll} 0^M(g) & = & 0, & (s+t)^M(g) & = & s^M(g) + t^M(g), \\ 1^M(g) & = & 1, & (s \cdot t)^M(g) & = & s^M(g) \cdot t^M(g), \\ x^M(g) & = & g(x), & \text{if } s^M \leq t^M, & \text{then} & \mu x s^M \leq \mu x t^M, \end{array}$$
2. $t^M(g) \leq t^M(h)$, if $g \leq h$ pointwise,
3. $t^M(g) = t^M(h)$, if $g = h$ on $\mathit{free}(t)$, (coincidence prop.)
4. $t[x/s]^M(g) = t^M(g[x/s^M(g)])$. (substitution prop.)

A Park μ-semiring is a partially ordered μ-semiring M where for all terms t and variables x, y, Park's axiom and rule hold in M:

$$t[x/\mu xt] \leq \mu xt, \qquad t[x/y] \leq y \to \mu xt \leq y.$$

In a Park μ-semiring M,

$$\mu xt^M(g) = \text{ the least } a \in M \text{ such that } t^M(g[x/a]) \leq a.$$

Grathwohl/Henglein/Kozen 2013: Chomsky-algebras

An idempotent semiring $(K, +, 0, \cdot, 1)$ is algebraically closed or a Chomsky-algebra, if every system

$$x_1 \geq p_1(\bar{x}, \bar{y}), \ldots, x_n \geq p_n(\bar{x}, \bar{y}), \quad \bar{x} = x_1, \ldots, x_n, \; n \in \mathbb{N},$$

with polynomials $p_i(\bar{x}, \bar{y})$ has a least solution $\bar{a} \in K^n$, for all parameters $\bar{b} \in K^m$ for $\bar{y} = y_1, \ldots, y_m$.

Example

The set $\mathcal{C}X^*$ of context-free languages over X form a Chomsky algebra $(\mathcal{C}X^*, \cup, \cdot, \emptyset, \{\epsilon\})$ – with incomplete partial order $\subseteq$.

Lemma (Grathwohl e.a. 2013)

Every Chomsky-algebra M is an idempotent, partially ordered μ-semiring, if we define for terms t, var. $x \in X$ and $g : X \to M$

$$\mu x t^M(g) := \text{the least } a \in M \text{ such that } t^M(g[x/a]) \leq a.$$

Moreover, every system $\bar{t}(\bar{x}, \bar{y}) \leq \bar{x}$ with μ-terms $\bar{t}(\bar{x}, \bar{y})$ has least solutions in M, i.e. for all parameters $\bar{b}$ from M there is a least tuple $\bar{a}$ in M such that $\bar{t}^M(\bar{a}, \bar{b}) \leq \bar{a}$.

Proof: by reduction to least solutions of polynomial systems.

Corollary

Every Chomsky algebra is a Park μ-semiring (using these $\mu x t^M$).

μ-Continuity

A Chomsky algebra M is μ-continuous, if for all $a, b \in M$, all terms t, $x \in X$ and $g : X \to M$ it satisfies

$$a \cdot \mu x t^M(g) \cdot b = \sum \{a \cdot mxt^M(g) \cdot b \mid m \in \mathbb{N}\}, \tag{1}$$

where mxt is defined by $0xt := 0$, $(m+1)xt := t[x/mxt]$.

The μ-continuity condition generalizes the *-continuity condition

$$a \cdot c^* \cdot b = \sum \{a \cdot c^m \cdot b \mid m \in \mathbb{N}\}.$$

Theorem (Completeness of $\text{Th}_=(\text{CFL})$, Grathwohl e.a. 2013)

For μ-terms s, t, the following are equivalent:

- $s^{\mathcal{P}X^*}(g) = t^{\mathcal{P}X^*}(g)$ *for the standard valuation* $g(x) = \{x\}$,
- $s^M(g) = t^M(g)$ *for all μ-continuous CAs M and $g : X \to M$.*

I. Every μ-continuous Chomsky algebra is a $\mathcal{C}$-dioid

We first define term vectors $\mu\bar{x}\bar{t}$ that embody H. Bekić's (1984) reduction of the n-ary least fixed-point operator to the unary one.

For vectors $\bar{t} = t_1, \ldots, t_n$ of terms and $\bar{x} = x_1, \ldots, x_n$ of pairwise different variables, define the term vector $\mu\bar{x}\bar{t}$ as follows.

If $n = 1$, then $\mu\bar{x}\bar{t} := \mu x_1 t_1$.

If $n > 1$, $\bar{x} = (\bar{y}, \bar{z})$ and $\bar{t} = (\bar{r}, \bar{s})$ with $|\bar{r}| = |\bar{y}| < n$, $|\bar{s}| = |\bar{z}| < n$,

$$\mu\bar{x}\bar{t} := (\mu\bar{y}.\bar{r}[\bar{z}/\mu\bar{z}\bar{s}], \mu\bar{z}.\bar{s}[\bar{y}/\mu\bar{y}\bar{r}]).$$

Lemma

For any Chomsky algebra M and valuation $g : X \to M$, $\mu\bar{x}\bar{t}^M(g)$ is the least $\bar{a} \in M^n$ such that $\bar{t}^M(g[\bar{x}/\bar{a}]) \leq \bar{a}$.

The value $\mu\bar{x}\bar{t}^M(g)$ does not depend on how $\bar{x}$ is split into $\bar{y}, \bar{z}$.

The unary version of μ-continuity implies the n-ary version:

Lemma

Let M be a μ-continuous Chomsky algebra and $g : X \to M$. Then

$$\bar{a} \cdot \mu\bar{x}\bar{t}^M(g) \cdot \bar{b} = \sum\{\bar{a} \cdot m\bar{x}\bar{t}^M(g) \cdot \bar{b} \mid m \in \mathbb{N}\},$$

for any term vector $\bar{t}$ and $\bar{a}, \bar{b} \in M^{|\bar{t}|}$, and $(m+1)\bar{x}\bar{t} := \bar{t}[\bar{x}/m\bar{x}\bar{t}]$.

Theorem

Let M be a μ-continuous Chomsky-algebra. Then M is a $\mathcal{C}$-dioid:

- *For each $U \in \mathcal{C}M$, a) $\sum U \in M$ exists.*
- *For all $U, V \in \mathcal{C}M$, b) $\sum(UV) = (\sum U)(\sum V)$.*

Proof. As M is a dioid, a) and b) are true for all $U, V \in \mathcal{F}M$.

Let $\bar{U} \in (\mathcal{C}M)^n$ be the least solution of $\bar{x} \geq \bar{p}^{\mathcal{C}M}(\bar{x}, \bar{A})$. By induction, we have a) and b) for all $U, V \in \bar{A} \cup \mathcal{F}M$, hence also

b') for all $a, b \in M$, $\sum aUb = a(\sum U)b$ and $\sum aVb = a(\sum V)b$

for all $U, V \in \bar{A}$. One can show that b') implies b).

To show a), b) for all $U, V \in \bar{U}, \bar{A}$, it therefore suffices that

a') Every $U \in \bar{U}$ has a supremum $\sum U \in M$.

b') For all $U \in \bar{U}$ and all $a, b \in M$, $\sum(aUb) = a(\sum U)b$.

By induction, we prove for $\bar{U}_m := m\bar{x}\bar{p}^{\mathcal{C}M}(\bar{A})$, $\bar{u}_m := m\bar{x}\bar{p}^M(\sum \bar{A})$

(i) $\sum(\bar{U}_m, \bar{A})$ exists (componentwise),

(ii) for all monomials $q(\bar{x}, \bar{y})$, $q^M(\sum \bar{U}_m, \sum \bar{A}) = \sum q^{\mathcal{C}M}(\bar{U}_m, \bar{A})$,

(iii) $\bar{u}_m = \sum \bar{U}_m$.

Notice (ii) is distributivity for $\sum$'s of sets in $\bar{U}, \bar{A}$.

For $m+1$, by induction $\sum \bar{U}_m$ exists by (i), and then

$$
\begin{aligned}
\bar{u}_{m+1} &= \bar{p}^M(\bar{u}_m, \textstyle\sum \bar{A}) && \text{(def.)} \\
&= \bar{p}^M(\textstyle\sum \bar{U}_m, \sum \bar{A}) && \text{by (iii)} \\
&= \textstyle\sum \bar{p}^{\mathcal{C}M}(\bar{U}_m, \bar{A}) && \text{by (ii)} \\
&= \textstyle\sum \bar{U}_{m+1} && \text{(def.)}
\end{aligned}
$$

Hence, (i) $\sum \bar{U}_{m+1}$ exists, and (iii) $\bar{u}_{m+1} = \sum \bar{U}_{m+1}$. (ii) is easy.

Now $\bar{u} := \mu\bar{x}\bar{p}^M(\sum \bar{A})$ is the least upper bound of $\bar{U} = \mu\bar{x}\bar{p}^{\mathcal{C}M}(\bar{A})$:

$$\begin{aligned}
\bar{u} &= \mu\bar{x}\bar{p}^M(\textstyle\sum \bar{A}) \\
&= \textstyle\sum\{m\bar{x}\bar{p}^M(\sum \bar{A}) \mid m \in \mathbb{N}\} && (M \text{ is } \mu\text{-continuous}) \\
&= \textstyle\sum\{\bar{u}_m \mid m \in \mathbb{N}\} \\
&= \textstyle\sum\{\sum \bar{U}_m \mid m \in \mathbb{N}\} && \text{(by (iii))} \\
&= \textstyle\sum \bar{U} = \sum \mu\bar{x}\bar{p}^{\mathcal{C}M}(\bar{A}).
\end{aligned}$$

In particular: a') each $U \in \bar{U}$ has a $\sum U \in M$.

To show b') $a(\sum U)b = \sum(aUb)$, extend a, b to some $\bar{a}, \bar{b} \in M^n$.

Having $\bar{a}(\sum \bar{U}_m)\bar{b} = \sum \bar{a}\bar{U}_m\bar{b}$ inductively by (ii), we obtain

$$\begin{aligned}
\bar{a}(\sum \bar{U})\bar{b} &= \bar{a} \cdot \bar{u} \cdot \bar{b} \\
&= \sum\{\bar{a} \cdot \bar{u}_m \cdot \bar{b} \mid m \in \mathbb{N}\} && (M\ \mu\text{-continuous}) \\
&= \sum\{\bar{a}(\sum \bar{U}_m)\bar{b} \mid m \in \mathbb{N}\} && (\bar{u}_m = \sum \bar{U}_m) \\
&= \sum\{\sum(\bar{a}U_m\bar{b}) \mid m \in \mathbb{N}\} && (\text{by (ii)}) \\
&= \sum \bigcup\{\bar{a}\bar{U}_m\bar{b} \mid m \in \mathbb{N}\} && (\textstyle\sum \text{ property}) \\
&= \sum(\bar{a} \cdot \bigcup\{\bar{U}_m \mid m \in \mathbb{N}\} \cdot \bar{b}) && (\cdot^{\mathcal{C}M} \text{ is } \textstyle\bigcup\text{-continuous}) \\
&= \sum(\bar{a}\bar{U}\bar{b}).
\end{aligned}$$

Hence, for $U \in \bar{U}$ we have b') $a(\sum U)b = \sum aUb$ for all a, b. □

II. Every $\mathcal{C}$-dioid is a μ-continuous Chomsky algebra

Theorem

Let M be a $\mathcal{C}$-dioid. Then M be a μ-continuous Chomsky-algebra.

Proof. (i) M is algebraically closed: For $\bar{x} \geq \bar{p}(\bar{x}, \bar{y})$ and $\bar{a} \in M^{|\bar{y}|}$, let $\bar{A}$ consist of the $A_j := \{a_j\} \in \mathcal{C}M$ and

$$\bar{U} := \mu\bar{x}\bar{p}^{\mathcal{C}M}(\bar{A}) \in (\mathcal{C}M)^{|\bar{x}|}$$

be the least solutuion of $\bar{x} \geq \bar{p}^{\mathcal{P}M}(\bar{x}, \bar{A})$ in $\mathcal{P}M$.

Since M is $\mathcal{C}$-complete, suprema $u_i := \sum U_i \in M$ exist.

Since M is $\mathcal{C}$-distributive, $\bar{u} = \sum \bar{U}$ is a solution of $\bar{x} \geq \bar{p}^M(\bar{x}, \bar{a})$:

$$\bar{p}(\bar{u}, \bar{a}) = \bar{p}^M(\sum \bar{U}, \sum \bar{A}) = \sum \bar{p}^{\mathcal{C}M}(\bar{U}, \bar{A}) \;\leq\; \sum \bar{U} = \bar{u}.$$

It is the least solution, since any solution $\bar{c}$ of $\bar{x} \geq \bar{p}^M(\bar{x}, \bar{a})$ is $\bar{c} > U$, since if $\bar{U}_m < \bar{c}$, also $\bar{U}_{m+1} = \bar{p}^{\mathcal{C}M}(\bar{U}_m, \bar{A}) < p^M(\bar{c}, \bar{a}) \leq \bar{c}$.

By (i), M is a Chomsky algebra, hence a Park μ-semiring under

$$\mu x t^M(g) := \text{the least } a \in M \text{ such that } t^M(g[x/a]) \leq a.$$

(ii) M is μ-continuous: we need an auxiliary

Claim. For all μ-terms $t(x_1, \ldots, x_n)$ and sets $A_1, \ldots, A_n \in \mathcal{C}M$,

$$t^M(\sum A_1, \ldots, \sum A_n) = \sum t^{\mathcal{C}M}(A_1, \ldots, A_n). \qquad (2)$$

Proof. By induction on t.

$$\begin{aligned}
(r \cdot s)^M(\sum \bar{A}) &= r^M(\sum \bar{A}) \cdot s^M(\sum \bar{A}) \\
&= (\sum r^{\mathcal{C}M}(\bar{A})) \cdot (\sum s^{\mathcal{C}M}(\bar{A})) \\
&= \sum (r^{\mathcal{C}M}(\bar{A}) \cdot s^{\mathcal{C}M}(\bar{A})) \quad (M \text{ is } \mathcal{C}\text{-distributive}) \\
&= \sum (r \cdot s)^{\mathcal{C}M}(\bar{A}).
\end{aligned}$$

For μxr, by induction we have for $B = \mu xr^{\mathcal{C}M}(\bar{A}) \in \mathcal{C}M$

$$r^M(\sum \bar{A}, \sum B) = \sum r^{\mathcal{C}M}(\bar{A}, B) \leq \sum B,$$

so that – by choice of μxr^M –

$$\mu xr^M(\sum \bar{A}) \leq \sum B = \sum \mu xr^{\mathcal{C}M}(\bar{A}).$$

The converse holds by induction on Kozen's well-ordering $\prec$ of μ-terms, assuming for all m

$$\sum mxr^{\mathcal{C}M}(\bar{A}) = mxr^M(\sum \bar{A}) \ \leq \ \mu xr^M(\sum \bar{A}).$$

We can now show the μ-continuity condition.

Claim. For all μ-terms $\mu x t(\bar{x})$, all $g : X \to M$ and $a, b \in M$:

$$a \cdot \mu x t^M(g) \cdot b = \sum \{a \cdot m x t^M(g) \cdot b \mid m \in \mathbb{N}\}.$$

Proof. Writing $g(x_i) = \sum A_i$ for $A_i = \{g(x_i)\} \in \mathcal{C}M$, we have

$$
\begin{aligned}
& a \cdot \mu x t^M(\sum \bar{A}) \cdot b \\
= \quad & (\sum\{a\})(\sum \mu x t^{\mathcal{C}M}(\bar{A}))(\sum\{b\}) && \text{(by (2))} \\
= \quad & \sum(\{a\} \cdot \mu x t^{\mathcal{C}M}(\bar{A}) \cdot \{b\}) && \text{(M is $\mathcal{C}$-distrib.)} \\
= \quad & \sum(\bigcup\{\{a\} \cdot m x t^{\mathcal{C}M}(\bar{A}) \cdot \{b\} \mid m \in \mathbb{N}\}) && \\
= \quad & \sum\{\sum(\{a\} \cdot m x t^{\mathcal{C}M}(\bar{A}) \cdot \{b\}) \mid m \in \mathbb{N}\} && \\
= \quad & \sum\{a \cdot (\sum m x t^{\mathcal{C}M}(\bar{A})) \cdot b \mid m \in \mathbb{N}\} && \text{(M is $\mathcal{C}$-distrib.)} \\
= \quad & \sum\{a \cdot m x t^M(\sum \bar{A}) \cdot b \mid m \in \mathbb{N}\}. && \text{(by (2))} \quad \square
\end{aligned}
$$

Prop. Let $f : M \to N$ be a semiring homomorphism between $\mathcal{C}$-dioids. Then f is an $\mathcal{C}$-morphism iff f preserves least solutions of polynomial inequations $\bar{x} \geq \bar{p}(\bar{x}, \bar{y})$.

Conclusion

The categories of $\mathcal{C}$-diods and μ-continuous Chomsky algebras agree.

Our proof by induction on the construction of $\mathcal{C}M$ shows a close connection between the computation of least solutions of

$$\bar{x} \geq \bar{p}^{M}(\bar{x}, \sum \bar{A}) \qquad \text{and} \qquad \bar{x} \geq \bar{p}^{\mathcal{C}M}(\bar{x}, \bar{A})$$

for polynomial systems $\bar{x} \geq \bar{p}(\bar{x}, \bar{y})$ and $\bar{A} \in \mathcal{C}M$.

Open Problems

1. Metalinear sets: There is a monadic operator $\mathcal{L} : \mathbb{M} \to \mathbb{DL}$, where $\mathcal{L}M$ is $\mathcal{F}M$ closed under $\cup$ and finite products of components of least solutions in $\mathcal{P}M$ of systems $\bar{x} \geq \bar{p}(\bar{x})$ with *linear* polynomials $\bar{p}(\bar{x})$ over $\mathcal{L}M$.
   - $\mathcal{L}$-dioid = linear-μ-continuous metalinear algebra ?
   - is the equational theory of metalinear languages completely axiomatized by: dioid-axioms + linear-μ-continuity axiom ?

2. To cover $\mathcal{S}M$ = context-sensitive subsets of M, consider the subcategory $\mathbb{M}_0$ of $\mathbb{M}$ with *non-erasing* homomorphisms only.
   (A_3): closure of CSL under "ϵ-free substitution".
   (A_4): image of a CSL under non-erasing homomor. is a CSL.

3. Construct an explicit adjunction $Q^{\mathcal{C}}_{\mathcal{R}} : \mathbb{D}\mathcal{R} \rightleftarrows \mathbb{D}\mathcal{C} : Q^{\mathcal{R}}_{\mathcal{C}}$ between *-continuous Kleene algebras and μ-continuous Chomsky algebras. ($Q^{\mathcal{C}}_{\mathcal{R}}$ by ideal-completion: Hopkins 2008.)

Alternative proof of Theorem 1

Lemma (Grathwohl e.a. 2013, Lemma 3.1)

Let M be a μ-continuous Chomsky algebra, $g : X \to M$ and $h : X \to \mathcal{C}X^*$ the canonical valuation $x \mapsto \{x\}$. Then for all μ-terms t and all $a, b \in M$ we have:

$$a \cdot t^M(g) \cdot b = \sum \{a \cdot w^M(g) \cdot b \mid w \in t^{\mathcal{C}X^*}(h)\}. \qquad (3)$$

Corollary: M is a $\mathcal{C}$-dioid.

Proof: Write $L_t := t^{\mathcal{C}X^*}(h)$, $U_t := \{w^M(g) \mid w \in L_t\}$.

Then $\mathcal{C}X^* = \{L_t \mid t \text{ a } \mu\text{-term }\}$, $\mathcal{C}M = \{U_t \mid t \text{ a } \mu\text{-term }\}$. By (3),

$$\sum U_t = t^M(g) \in M \quad \text{and} \quad a(\sum U_t)b = \sum(aU_tb). \quad \square$$