Counting Finite Linearly Ordered Involutive Bisemilattices

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Joint work with
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- Involutive Bisemilattices;
  - Weak Kleene Logic;
  - Płonka Sums of Boolean Algebras;
  - Duality for IBSLs;
- Linearly ordered Involutive Bisemilattices;
  - Structural Properties;
  - Algorithm for Counting L-IBSLs;
- Conclusions;
  - Results and Comparisons;
  - Further Developments and Links;

An **involutive bisemilattice** is an algebra $\mathbf{B} = \langle B, \vee, \wedge, \neg, 0, 1\rangle$ of type $(2, 2, 1, 0, 0)$ satisfying:

- I1 $x \vee x \approx x$;
- I2 $x \vee y \approx y \vee x$;
- I3 $x \vee (y \vee z) \approx (x \vee y) \vee z$;
- I4 $\neg(\neg x) \approx x$;
- I5 $x \wedge y \approx \neg(\neg x \vee \neg y)$;
- I6 $x \wedge (\neg x \vee y) \approx x \wedge y$;
- I7 $0 \vee x \approx x$;
- I8 $1 \approx \neg 0$.

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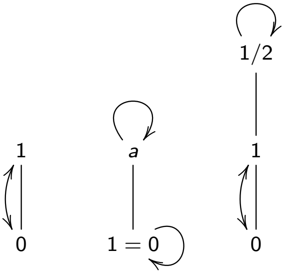

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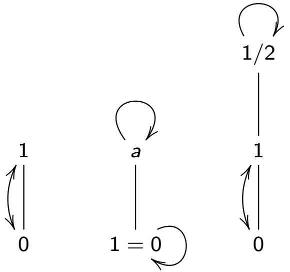

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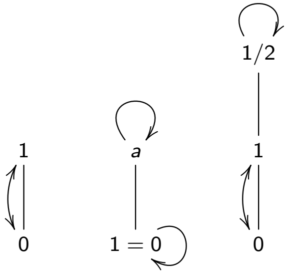

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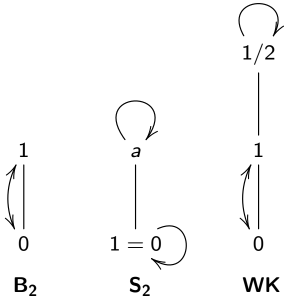

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The variety $\mathcal{IBSL}$ is the **regularization** of the variety $\mathcal{BA}$, of Boolean algebras, i.e. $\mathcal{IBSL} \models \varepsilon \approx \tau$ if and only if $\mathcal{BA} \models \varepsilon \approx \tau$ and $Var(\varepsilon) = Var(\tau)$.

$\mathcal{IBSL}$ is the *algebraic counterpart* of the **Weak Kleene Logic**:

$\wedge$	0	$\frac{1}{2}$	1
0	0	$\frac{1}{2}$	0
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
1	0	$\frac{1}{2}$	1

$\vee$	0	$\frac{1}{2}$	1
0	0	$\frac{1}{2}$	1
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
1	1	$\frac{1}{2}$	1

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$\vee$	0	$\frac{1}{2}$	1
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- Beall interpretation (2016): $\frac{1}{2}$ is **off-topic**, 1 is *true and on-topic*, 0 is *false and on-topic*;

Let $\mathbb{C}$ be an arbitrary category.
A **direct system** in $\mathbb{C}$ is a triple $\mathbb{X} = \langle X_i, p_{ii'}, I \rangle$ such that

- I is a semilattice.
- $\{X_i\}_{i \in I}$ forms an indexed family of objects in $\mathbb{C}$ with disjoint universes;
- $p_{ii'} : X_i \to X_{i'}$ is a morphism of $\mathbb{C}$, for each pair $i \leq i'$ $(i, i' \in I)$, satisfying that p_{ii} is the identity in X_i and such that $i \leq i' \leq i''$ implies $p_{i'i''} \circ p_{ii'} = p_{ii''}$.

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A morphism between two semilattice direct systems $\mathbb{X} = \langle X_i, p_{ii'}, I \rangle$ and $\mathbb{Y} = \langle Y_j, q_{jj'}, J \rangle$ is a pair $(\varphi, \{f_i\}_{i \in I}) \colon \mathbb{X} \to \mathbb{Y}$ such that

i) $\varphi \colon I \to J$ is a semilattice homomorphism

ii) $f_i \colon X_i \to Y_{\varphi(i)}$ is a morphism in $\mathbb{C}$, making the following diagram commutative, for each $i \leq i' \in I$:

$$\begin{array}{ccc} X_i & \xrightarrow{p_{ii'}} & X_{i'} \\ \downarrow{\scriptstyle f_i} & & \downarrow{\scriptstyle f_{i'}} \\ Y_{\varphi(i)} & \xrightarrow{q_{\varphi i \varphi i'}} & Y_{\varphi(i')} \end{array}$$

Let $\mathbb{A} = \langle \mathbf{A}_i, p_{ii'}, I \rangle$ be a semilattice direct system of algebras (with signature ν).

The **Płonka sum** over $\mathbb{A}$ is the algebra $\mathcal{P}_I(\mathbb{A}) = \langle \bigcup_{i \in I} A_i, \{g^{\mathcal{P}_I(\mathbf{A}_i)} : g \in \nu\} \rangle$

- for every n-ary $g \in \nu$ ($n \geq 1$), and $a_1, \ldots, a_n \in \bigcup_I A_i$
  $a_1 \in A_{i_1}, \ldots, a_n \in A_{i_n}$
  Set $j = i_1 \vee \cdots \vee i_n$

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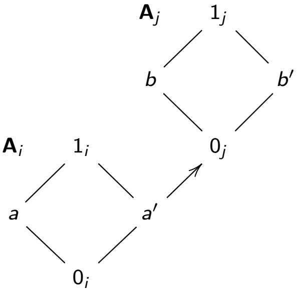

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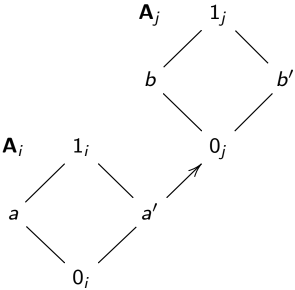

- $a \wedge^{\mathcal{P}_I} a' = a \wedge^{\mathbf{A}_i} a' = 0_i$,
- $a' \wedge^{\mathcal{P}_I} b = p_{ij}(a') \wedge^{\mathbf{A}_j} b = 0_j \wedge^{\mathbf{A}_j} b = 0_j$.

Theorem (Bonzio, Gil-Férez, Paoli, Peruzzi. 2017)

If $\mathbb{A}$ is a semilattice direct system of Boolean algebras, then $\mathcal{P}_l(\mathbb{A})$ is an involutive bisemilattice, and if $\mathbf{B}$ is an involutive bisemilattice, then $\mathbf{B} \cong \mathcal{P}_l(\mathbb{A})$, where $\mathbb{A}$ is a semilattice direct system of Boolean algebras.

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*The finite slice of $\mathcal{IBSL}$ is dually equivalent to Sem-inv-*Set.

Where *Sem-inv*-$\mathbb{C}$ is the category of semilattice **inverse** systems of $\mathbb{C}$, obtained analogously at Sem-dir-$\mathbb{C}$, by, intuitively, reversing the directions of transition morphisms.

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$\mathcal{L} - \mathcal{IBSL}$ seems to be a good class to study before approaching $\mathcal{IBSL}$:

- $\mathcal{L} - \mathcal{IBSL}$ is smaller and easier to handle than $\mathcal{IBSL}$;
- it has nice combinatorial/structural properties;
- finite L-IBSLs are generated from **binary partitions**.

Properties

Let $\mathbb{A} = \langle \mathbf{A}_i, p_{ii'}, I \rangle$ and $\mathbb{B} = \langle \mathbf{B}_i, q_{ii'}, I \rangle$ be two finite semilattice direct systems of Boolean algebras, with I linearly ordered and **containing no trivial algebras**. Then, the following statements are equivalent:

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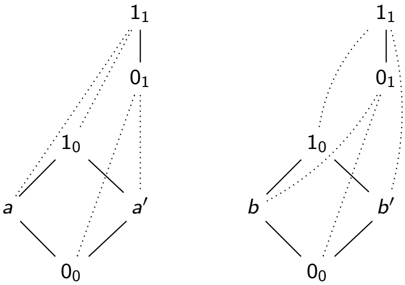
2. $\mathbf{A}_i \cong \mathbf{B}_i$, for every $i \in I$, and $| \widehat{A}_{i'} / ker(\widehat{p}_{ii'}) | = | \widehat{B}_{i'} / ker(\widehat{q}_{ii'}) |$, for every $i < i'$.



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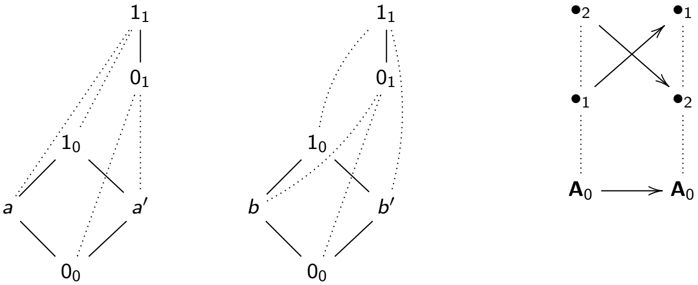

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Let $\mathbb{A}_{p_{01}} = \langle \{\mathbf{A}_0, \mathbf{A}_1\}, p, \{0 < 1\}\rangle$ be a family of linearly ordered semilattice direct system of Boolean algebras. The number of non-isomorphic involutive bisemilattices obtained over $\mathbb{A}_{p_{01}}$ is the number of non-isomorphic involutive bisemilattices of cardinality $| A_0 | + | A_1 |$ and is equal to

$$N(\mathbb{A}_{p_{01}}) := N(A_0, A_1) = \min(| \widehat{A_0} |, | \widehat{A_1} |),$$

where $\widehat{A_0}$ ($\widehat{A_1}$, resp.) is the dual space of $\mathbf{A}_0$ ($\mathbf{A}_1$, resp.).

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Let $\langle\{\mathbf{A}_0, \ldots, \mathbf{A}_m\}, p_{i-1,i}, I\rangle$ be a family of semilattice direct systems of Boolean Algebras. Then,

$$N(\mathbb{A}_{p_{0m}}) = N(\mathbb{A}_{p_{01}}) \cdot N(\mathbb{A}_{p_{12}}) \cdot \ldots \cdot N(\mathbb{A}_{p_{m-1m}}) = \prod_{i=0}^{m-1} N(\mathbb{A}_{p_{ii+1}}).$$

Let $n \in \mathbb{N}^+$, and $e, m_i \in \mathbb{N}$ for $0 \leq i \leq e$.
A **binary partition** of n is a decomposition of n into powers of two, that is

$$n = m_e \cdot 2^e + m_{e-1} \cdot 2^{e-1} + \cdots + m_0 \cdot 2^0.$$

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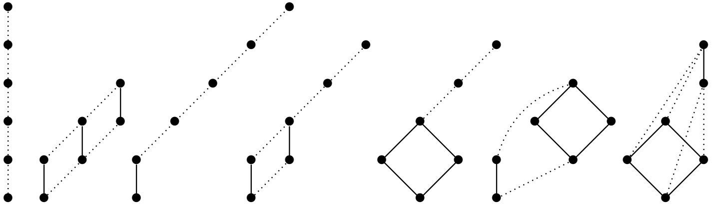
Each finite linearly ordered $\mathcal{IBSL}$ $\mathbf{B}$ can be decomposed as a Płonka sum of Boolean algebras $\mathbf{A}_i$ whose direct system is indexed by a totally ordered set I.
Then, it follows that the cardinality n of $\mathbf{B}$ is always given by a solution of a binary partition where $2^i = |\mathbf{A}_i|$ and $|I| = \sum_{i=0}^{e} m_i$.

The 7 linearly ordered non-isomorphic IBSLs of cardinality 6.



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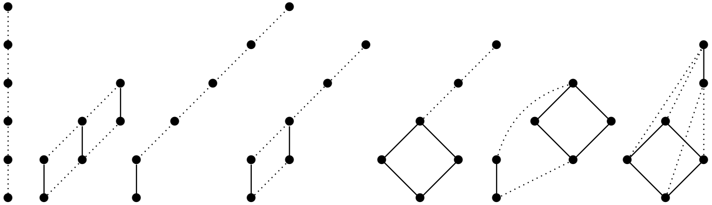

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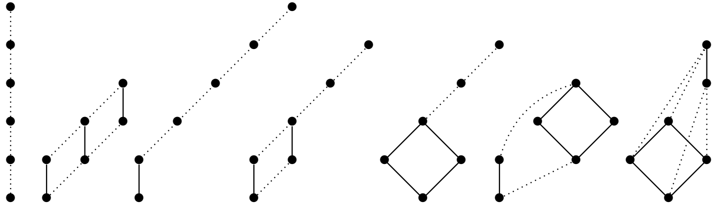

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Partition 6 generates **two** L-IBSLs (we have to consider *permutations with repetitions*).

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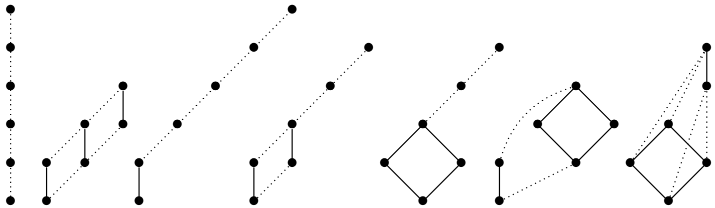

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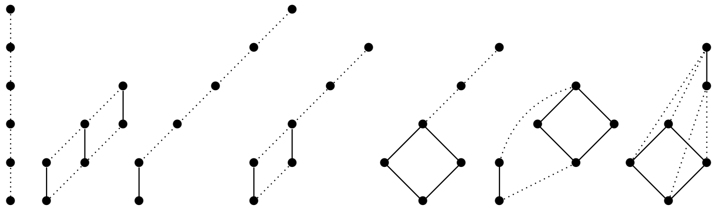
1. $(6, 2^0)$;
2. $(3, 2^1)$;
3. $(1, 2^1) \to (4, 2^0)$;
4. $(2, 2^1) \to (2, 2^0)$;
5. $(1, 2^2) \to (2, 2^0)$;
6. $(1, 2^2) \to (1, 2^1)$;

Partition 6 generates **two** L-IBSLs (we have to consider *permutations with repetitions*).

The 7 linearly ordered non-isomorphic IBSLs of cardinality 6.



1. $1 \cdot 2^0 + 1 \cdot 2^0 + 1 \cdot 2^0 + 1 \cdot 2^0 + 1 \cdot 2^0 + 1 \cdot 2^0$;
2. $1 \cdot 2^1 + 1 \cdot 2^1 + 1 \cdot 2^1$;
3. $1 \cdot 2^1 + 1 \cdot 2^0 + 1 \cdot 2^0 + 1 \cdot 2^0 + 1 \cdot 2^0$;
4. $1 \cdot 2^1 + 1 \cdot 2^1 + 1 \cdot 2^0 + 1 \cdot 2^0$;
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Partition 6 generates **two** L-IBSLs (we have to consider *permutations with repetitions*). Sequences 5 and 6 are *branches* of the same *tree* with root $(1, 2^2)$.

We call $\textsc{GenForest}(n)$ the *function* that computes the set of trees corresponding to the binary partitions $b(n)$.

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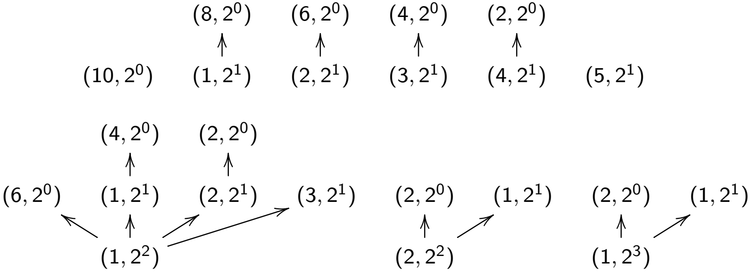
For $n = 10$ the output of $\textsc{GenForest}(10)$ is:



Counting

```
function L-IBSL(n)
    F = GenForest(n)
    for each tree T in F do
        for each branch B in T do            ▷ Notice that B is a list of couples (i, 2^e)
            B' = B without all the couples (i, 2^0)
            for each permutation P of B' do
                for each (i, 2^{e_i}) → (j, 2^{e_j}) in P do
                    t = t×N(2^{e_i}, 2^{e_j})
                end for
            end for
        end for
    end for
    return t
end function
```

```
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                    t = t × N(2^{e_i}, 2^{e_j})
                end for
            end for
        end for
    end for
    return t
end function
```

Theorem

The number of all the non-isomorphic $\mathcal{L}-\mathcal{IBSL}$ of cardinality n is given by $\mathrm{L\text{-}IBSL}(n)$.

Runnings done on a GNU/Linux Debian 4.9.82-1 system with an Intel Core i7-5500U CPU and 8GB of RAM.

The running times of our experiments as calculated by the tool `time`.

Running Times	Algorithm	Mace4	`interpformat`	`isofilter`
`real`	0m1.331s	1m1.115s	0m40.891s	0m43.611s
`user`	0m1.176s	1m0.008s	0m40.484s	0m43.524s
`sys`	0m0.156s	0m0.908s	0m0.232s	0m0.068s

The second column reports the total time used by the Python implementation of our algorithm to count all the non-isomorphic LIBSLs of cardinality $1 \leq n \leq 23$.
The third column reports the time used by Mace4 to generate the first-order models of cardinality $2 \leq n \leq 11$.
The fourth column reports the time used by `interpformat` to transform the Mace4 models in a format useful for `isofilter`.
The fifth column reports the time used by `isofilter` to produce a file with all the non-isomorphic LIBSLs of cardinality $2 \leq n \leq 11$.

The numbers of L-IBSL with n elements for $1 \leq n \leq 23$.

n	1	2	3	4	5	6	7	8
L-IBSL(n)	1	2	2	4	4	7	7	14
n	9	10	11	12	13	14	15	16
L-IBSL(n)	14	26	26	52	52	99	99	199
n	17	18	19	20	21	22	23	...
L-IBSL(n)	199	386	386	772	772	1508	1508	...

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Future Works:

- improving the algorithm and the implementation
  [non-recursive GenForest, C/C++ instead of Python, ...];
- extending the approach to $\mathcal{IBSL}$
  [by using (Heitzig and Reinhold, 2002) to generate finite (semi-)lattices];
- analysis of the algorithms / complexity;
- tackle *fine spectrum* problem (Taylor, 1975) for other varieties (via dualities);
  when dealing with ordered structures, *"the fine spectrum problem is usually hopeless"* (Quackenbush, 1982).