On the structure of generalized effect algebras and separation algebras

Peter Jipsen
joint work with Sarah Alexander and Nadiya Upegui

Chapman University

Ordina, Groningen, the Netherlands
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Outline

- Partial Algebras
- Separation algebras
- Generalized effect algebras
- Constructing all GPE-algebras of size n
- Some theorems obtained from this output

What is a Partial Algebra?

- A *partial operation* g of arity n on a set A is a function from a subset $D(g)$ of A^n to A.
- The notation $g : A^n \multimap A$ is used to indicate that g is an n-ary partial function on A.
- A *partial algebra* is a pair $\mathbf{A} = (A, \mathcal{F}^{\mathbf{A}})$ where A is a set and $\mathcal{F}^{\mathbf{A}}$ is a set of operations on A containing at least one partial operation.

$+$	0	1	2	3
0	0	1	2	3
1	1	2	3	-
2	2	3	-	-
3	3	-	-	-

Separation Algebras

A **separation algebra** (or SA) $\mathbf{A} = (A, +, 0)$ is a partial algebra such that for all $x, y, z \in A$

$$
\begin{aligned}
&\text{(canc)} \quad x + y \text{ defined and } x + y = x + z \implies y = z \\
&\text{(comm)} \quad x + y \text{ defined} \implies x + y = y + x \\
&\text{(asso)} \quad (x + y) + z \text{ defined} \implies (x + y) + z = x + (y + z) \\
&\text{(iden)} \quad x + 0 = x
\end{aligned}
$$

In short, an SA is a **cancellative commutative partial monoid**.

Separation algebras are **naturally pre-ordered** by

$$x \leq y \quad \iff \quad \exists w \; x + w = y$$

Any abelian group is a (total) separation algebra ($\leq$ relates all elements).

Generalized Effect Algebras

$(\mathbb{N}, +, 0)$ is another (total) separation algebra.

A **generalized effect algebra** (or GEA) $\mathbf{A} = (A, +, 0)$ is a separation algebra such that for all $x, y \in A$ we have

(positivity) $x + y = 0 \implies x = 0 = y$

GEAs are natually **partially** ordered by

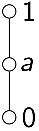
$$x \leq y \quad \iff \quad \exists w \; x + w = y$$

An **effect algebra** is a GEA with a top element, denoted 1.

Can define x' by $y = x' \iff x + y = 1$.

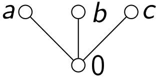
Examples of GE-Algebras

An effect algebra of size 3



$+$	0	a	1
0	0	a	1
a	a	1	–
1	1	–	–

A GEA of size 4



$+$	0	a	b	c
0	0	a	b	c
a	a	–	–	–
b	b	–	–	–
c	c	–	–	–

Why Study Effect Algebras?

Effect algebras have applications in the foundations of **quantum mechanics** and in **probability theory**.

D. J. Foulis and M. K. Bennett [1994]:

If a quantum-mechanical system $\mathcal{S}$ is represented in the usual way by a Hilbert space $\mathcal{H}$, then a self-adjoint operator A on $\mathcal{H}$ such that $\mathbf{0} \leq \mathbf{A} \leq \mathbf{1}$ corresponds to an **effect** for $\mathcal{S}$. Effects are of significance in representing **unsharp** measurements or observations on the system $\mathcal{S}$, and effect valued measures play an important role in stochastic quantum mechanics.

Why Study Separation Algebras?

Let $\mathbf{A}$ be a separation algebra and for $X, Y \subseteq A$ define $X * Y = \{x + y \mid x \in X, y \in Y\}$, the complex lifting of $+$.

The complex algebra $(\mathcal{P}(A), \cup, \cap, \neg, \emptyset, A, *, -\!\!*, \{0\})$ is a complete and atomic Boolean algebra with a separating conjunction $*$ and a residual $X -\!\!* Y = \{z \in A \mid X * \{z\} \subseteq Y\}$.

This is a **Boolean bunched implication algebra**.

In logical form, Boolean bunched implication logic is used in **separation logic** to reason about pointer structures and concurrency of programs.

Concrete examples of separation algebras arise from modeling a memory heap as partial functions f from $\mathbb{N}$ (addresses) to V (values).

$f * g$ is defined and $= f \cup g \quad \iff \quad D(f) \cap D(g) = \emptyset$.

Generalized SAs and Generalized Pseudo EAs

Generalized separation algebras are cancellative partial monoids with **conjugation**: $\exists z(x+z=y) \iff \exists w(w+x=y)$

This axiom ensures that there is only **one** natural pre-order.

A generalized pseudo effect algebra (GPEA) is a positive GSA

$\mathbf{A} = (A, +, 0)$

$+$	0	a	b	c	1
0	0	a	b	c	1
a	a	$-$	1	$-$	$-$
b	b	$-$	$-$	1	$-$
c	c	1	$-$	$-$	$-$
1	1	$-$	$-$	$-$	$-$

This is a GPE-algebra.

$\mathbf{B} = (A \setminus \{c\}, +{\restriction_B}, 0)$

$+{\restriction_B}$	0	a	b	1
0	0	a	b	1
a	a	$-$	1	$-$
b	b	$-$	$-$	$-$
1	1	$-$	$-$	$-$

This is a closed subalgebra of **A** that fails conjugation.

Downward Closed Subsets of GPE-Algebras

Lemma

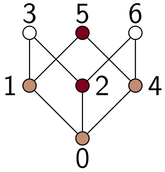
Let $\mathbf{A} = (A, +, 0)$ *be a GPE-algebra and* $0 \in B \subseteq A$.

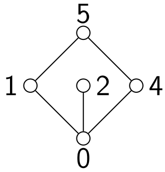
Define the downward closed subset $\downarrow B$ *of* B *by*

$\downarrow B := \{x \in A \mid x \leq y \text{ for some } y \in B\}$.

Then $\mathbf{B} = (\downarrow B, +\restriction_{\downarrow B}, 0)$ *is a GPE-algebra.*

Example of a Downward Closed Subset





+	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	—	3	—	5	—	—
2	2	3	—	—	6	—	—
3	3	—	—	—	—	—	—
4	4	5	6	—	—	—	—
5	5	—	—	—	—	—	—
6	6	—	—	—	—	—	—

+	0	1	2	4	5
0	0	1	2	4	5
1	1	—	—	5	—
2	2	—	—	—	—
4	4	5	—	—	—
5	5		—	—	—

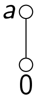
Products of GPE-aglebras (continued)

Lemma

The direct product of a family of GPE-algebras is also a GPE-algebra.

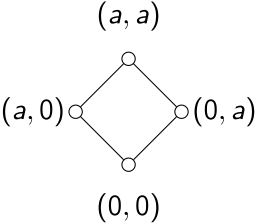
A

$+$	0	a
0	0	a
a	a	$-$



A × A

$+$	$(0,0)$	$(0,a)$	$(a,0)$	(a,a)
$(0,0)$	$(0,0)$	$(0,a)$	$(a,0)$	(a,a)
$(0,a)$	$(0,a)$	$-$	(a,a)	$-$
$(a,0)$	$(a,0)$	(a,a)	$-$	$-$
(a,a)	(a,a)	$-$	$-$	$-$



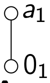
Simple Pastings of GPE-algebras (continued)

Lemma

The simple pasting of a family of GPE-algebras is also a GPE-algebra.

$\mathbf{A}_1$

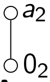
$+_1$	0_1	a_1
0_1	0_1	a_1
a_1	a_1	$-$



$\mathbf{A}_1$

$\mathbf{A}_2$

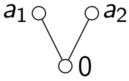
$+_2$	0_2	a_2
0_2	0_2	a_2
a_2	a_2	$-$



$\mathbf{A}_2$

$\mathbf{A}_1 +_0 \mathbf{A}_2$

$+$	0	a_1	a_2
0	0	a_1	a_2
a_1	a_1	$-$	$-$
a_2	a_2	$-$	$-$

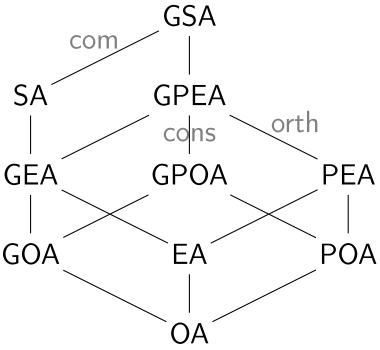


$\mathbf{A}_1 +_0 \mathbf{A}_2$

Subclasses and Expansions of GPE-algebras

Adding combinations of three independent axioms creates subclasses:

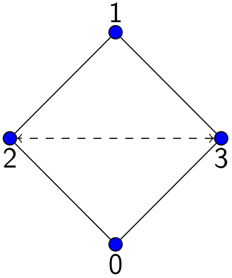
(com) $x + y = y + x$ (**commutative**)
(orth) $x + y = 1 \iff y = x^{\sim} \iff x = y^{-}$ (**orthocomplement**)
(cons) $x + x$ defined $\implies x = 0$ (**consistent**)



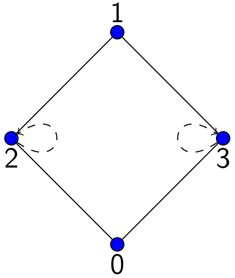
G = Generalized, S = Separation, P = Pseudo, E = Effect, O = Ortho

Examples

Orthoalgebra
(Consistency Satisfied)

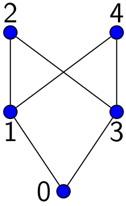


Effect Algebra
(Consistency Not Satisfied)

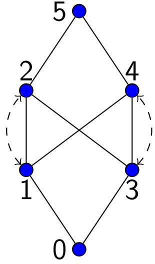


Examples

Generalized Effect Algebra
(Orthocomplementation Not Satisfied)

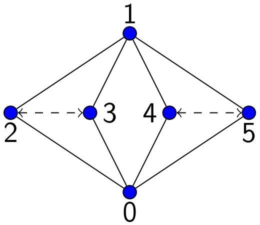


Effect Algebra
(Orthocomplementation Satisfied)

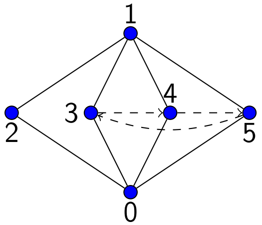


Examples

Orthoalgebra
(Commutativity Satisfied)



Pseudo Orthoalgebra
(Commutativity Not Satisfied)



From Separation Algebras to Effect Algebras

An element v is *invertible* if there exists w such that $vw = e = wv$

A^* denotes the set of invertible elements of a GS-algebra $\mathbf{A}$.

The inverse of v, if it exists, is unique and is denoted by v^{-1}.

Lemma

Let $\mathbf{A}$ *be a generalized separation algebra. Then*

1. A^* *is the bottom equivalence class* $[e]$ *of the poset* $A/{\equiv} = (\{[x] : x \in A\}, \leq)$,
2. $\mathbf{A}^* = (A^*, \cdot, e, {}^{-1})$ *is a (total) group and is a closed subalgebra of* $\mathbf{A}$,
3. $x \equiv y$ *holds if and only if* $x \in yA^*$, *and*
4. $\equiv$ *is the identity relation if and only if* e *is the only invertible element.*

From Separation Algebras to Effect Algebras

Every separation algebra can be collapsed in a unique way to a largest generalized effect algebra.

Hence a substantial part of the structure theory of separation algebras is covered by results about generalized effect algebras.

Theorem

For a GS-algebra $\mathbf{A}$,

1. *the relation* $\equiv$ *is a closed congruence,*
2. $\mathbf{A}/\!\equiv$ *is a GPE-algebra,*
3. *the congruence classes of* $\equiv$ *all have the same cardinality, and*
4. *if* $h : \mathbf{A} \to \mathbf{B}$ *is a homomorphism and* $\mathbf{B}$ *is a GPE-algebra then there exists a unique homomorphism* $g : \mathbf{A}/\!\equiv\, \to \mathbf{B}$ *such that* $g \circ \gamma = h$ *(where* $\gamma : \mathbf{A} \to \mathbf{A}/\!\equiv$ *is the canonical homomorphism* $\gamma(x) = [x]$*).*

From abelian groups and effect algebras to separation algebras

Theorem

Let $\mathbf{G}$ be an abelian group and $\mathbf{B}$ a GE-algebra.
Then $\mathbf{A} = \mathbf{G} \times \mathbf{B}$ is a separation algebra with $\mathbf{A}^* = \mathbf{G} \times \{e\}$.

Similarly the product of a group and a GPE-algebra is a GS-algebra.

Proof.

The product of separation algebras is again a separation algebra since this class of algebras is defined by quasi-identities.
The element $(g, e) \in A$ has inverse (g^{-1}, e).
Now let $b \in B$. If (g, b) has an inverse (h, c) then $bc = e$, hence by positivity of B we have $b = e$.
Therefore $\mathbf{A}^* = \mathbf{G} \times \{e\}$. □

Building GPE-algebras

Theorem

Let $\mathbf{P} = (P, \oplus, 0)$ *be a GPEA. Let* $P_m = P \cup \{m\}$ *where* $m \notin P$*. Then* $\mathbf{P}_m = (P_m, +, 0)$ *is a GPEA if and only if the following conditions hold:*

(1) *For all* $x, y \in P$, $x + y \in P$ *iff* $x \oplus y$ *is defined, in which case* $x + y = x \oplus y$.

(2) $m + 0 = m = 0 + m$

(3) $m + x$ *and* $x + m$ *are undefined for all* $x \in P_m \setminus \{0\}$

(4) $x + y = m = x + z \implies y = z$ *and* $x + y = m = z + y \implies x = z$

(5) *For all* $x, y \in P$, $x + y = m \implies \exists u, v$ *s.t.* $u + x = m = y + v$

(6) $(x + y) + z = m \iff x + (y + z) = m$

Enumerating GPE-algebras: Initial Setup

Consider a GPE-algebra $\mathbf{P} = (P, \oplus, 0)$.

The program generates a new GPE-algebra $\mathbf{P_m} = (P_m, +, 0)$, with $P_m = P \cup \{m\}$.

Initial rules for $+$:

(1) For all $x, y \in P$, $x + y$ is defined iff $x \oplus y$ is defined, in which case $x + y = x \oplus y$ (**satisfies posi**)

(2) $m + 0 = m = 0 + m$ (**satisfies iden**)

(3) For all $x \in P_m$ such that $x \neq 0$, $x + m$ and $m + x$ are undefined

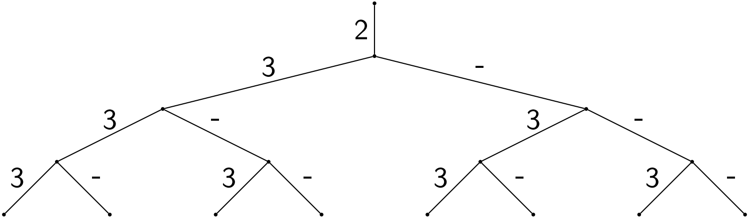
(4) For all $x, y \in P$ such that $x \oplus y$ is undefined, $x + y$ is not yet determined, which will be represented by $x + y = N$

Process of filling out the operation table

$\oplus$	0	1	2
0	0	1	2
1	1	2	-
2	2	-	-

$\rightarrow$

$+$	0	1	2	3
0	0	1	2	3
1	1	2	N	-
2	2	N	N	-
3	3	-	-	-



Checking for cancellativity, conjugation and associativity

A table is **cancellative** if each element appears no more than once in every row/column.

Cancellative

+	0	1	2	3
0	0	1	2	3
1	1	2	-	-
2	2	-	3	-
3	3	-	-	-

Not Cancellative

+	0	1	2	3
0	0	1	2	3
1	1	2	3	-
2	2	-	3	-
3	3	-	-	-

$$1 + 2 = 3$$
$$2 + 2 = 3$$

Checking for Conjugation

A table is **conjugative** if for all i, j:

- each element defined in row i is also defined in column i
- each element defined in column j is also defined in row j.

Conjugative

+	0	1	2	3
0	0	1	2	3
1	1	2	3	-
2	2	3	-	-
3	3	-	-	-

Not Conjugative

+	0	1	2	3
0	0	1	2	3
1	1	2	3	-
2	2	-	-	-
3	3	-	-	-

$$1 + 2 = 3$$
$$\nexists u(u + 1 = 3)$$
$$\nexists v(2 + v = 3)$$

Checking for Associativity

A table is **associative** if for all $x, y, z \in P$:

- If $(x + y) + z$ is undefined, then $x + (y + z)$ is also undefined.
- If $(x + y) + z$ is defined, then $(x + y) + z = x + (y + z)$.

```
for all x,y,z in P_m where (x+y) is defined:
    if (x+y)+z is undefined:
        if y+z and x+(y+z) are defined:
            return False
    if (x+y)+z is defined:
        if y+z or x+(y+z) are undefined:
            return False
        if x+(y+z) != (x+y)+z:
            return False
return True
```

Counting (G)(P)(O) Effect algebras and Separation algebras

n	OA	POA	GOA	EA	PEA	GPOA	GEA	SA	GPEA	GSA
2	1	1	1	1	1	1	1	2	1	2
3	0	0	1	1	1	1	2	3	2	3
4	1	1	2	3	3	2	5	8	5	8
5	0	1	2	4	5	3	12	13	13	14
6	1	2	4	10	12	7	35	39	42	48
7	0	2	8	14	19	19	119	120	171	172
8	2	5	18	40	52	68	496	507	1020	1037
9	0	4	42	60	84	466	2699	2703	11742	11749
10	2	10	156	172	240	8740	21888	21905	322918	
11	0	9	834	282	418		292496	292497		

Table: Number of partial algebras in each class

O = Ortho, P = Pseudo, G = Generalized, E = Effect, S = Separation

Further results about GPE-algebras

The **height** of an element a in a finite GPE-algebra is the length of a maximal path from 0 to a in the Hasse diagram of the partial order.

A set of elements of the same height make up a **level**.

The **atoms** of a GPE-algebra are the elements in level 1, i.e, they only have the bottom element 0 below them.

Lemma

Associativity holds automatically for naturally ordered partial algebras that have two levels or less.

Lemma

A GPE-algebra is a GE-algebra if and only if it has a generating set in which all elements commute.

Further results about GPE-algebras

Recall that every 1-generated group is commutative.

Theorem

Every 1- or 2-generated GPE-algebra is commutative.

Let $L(n_1, n_2, ..., n_k)$ denote the number of GPE-algebras (up to isomorphism) with level structure $(n_1, n_2, ..., n_k)$ and $n = 1 + \sum_{i=1}^{k} n_i$ number of elements.

The number of **integer partitions** $p(n)$ for a positive integer n is the number of ways positive integers can sum to n, ignoring order.

We now show that the number of GPE-algebras of height ≤ 2 with cardinality n is given by the sum of $p(k)$ for $k = 1$ to $n - 2$.

Further results about GPE-algebras

A partial operation $+$ can be viewed as a coalgebra $\alpha : A \to \mathcal{P}(A^2)$ where $\alpha(x) = \{(y, z) \in A^2 \mid x = y + z\}$.

Lemma

For a GPE-algebra $\mathbf{A}$ *and* $x \in A$*, the binary relation* $\alpha(x)$ *is a permutation of its domain, hence in the finite case the domain is partitioned into disjoint finite cycles.*

Lemma

For any GPE-algebra of size $n \geq 3$*,* $L(n-2, 1) = L(n-3, 1) + p(n-2)$*.*

Theorem

The number of GPE-algebra of cardinality n *with level structure* $(n-2, 1)$ *is* $\sum_{k=1}^{n-2} p(k)$*.*

Residuated posets from GPE-algebras

A **residuated partially ordered monoid** $(A, \leq, \cdot, e, \backslash, /)$ is a poset $(A, \leq)$, a monoid $(A, \cdot, e)$, and for all $x, y, z \in A$, $xy \leq z \Leftrightarrow y \leq x\backslash z \Leftrightarrow y \leq z/x$.

Definition

Let $\mathbf{A} = (A, +, 0)$ be a generalized pseudo-effect algebra.
Define $\bar{\mathbf{A}} = (A \cup \{\bot, \top\}, \cdot, e, \backslash, /)$ as follows:

- $e = 0$ and $\bot < x < \top$ for any $x \in A$
- $x \cdot y := x + y$ if $x + y$ is defined, else $x \cdot y = \top$ for $x, y \in A$
- $y/x = z \iff y = z + x$ and $x\backslash y = z \iff y = x + z$
- $y/x = x\backslash y = \bot$ if $x \not\leq y$
- $\bot/x = x\backslash\bot = x/\top = \top\backslash x = \bot$
- $\bot x = x\bot = \bot$ and $y\top = \top y = \top$ for $x, y \in A \cup \{\bot, \top\}$ with $y \neq \bot$

Residuated posets from GPE-algebras

Theorem (Rump, Yang 2014)

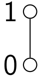
Let $\mathbf{A}$ *be a GPE-algebra. Then* $\bar{\mathbf{A}}$ *is a residuated poset.*

Corollary

Every GPE-algebra is an interval in some (total) residuated poset.
A GPE-algebra $\mathbf{A}$ *is lattice-ordered* $\iff$ $\bar{\mathbf{A}}$ *is a residuated lattice.*

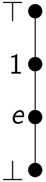
E_2

$\oplus$	0	1
0	0	1
1	1	–



$\bar{E}_2$

$\cdot$	$\top$	1	e	$\bot$
$\top$	$\top$	$\top$	$\top$	$\bot$
1	$\top$	$\top$	1	$\bot$
e	$\top$	1	e	$\bot$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$



Pseudo-effect algebras and involutive residuated lattices

It is also possible to axiomatize the residuated po-monoids that uniquely correspond to GPE-algebras

A residuated poset is **involutive** if there exists an element d such that the terms $\sim x = x \backslash d$ and $-x = d/x$ satisfy $-\sim x = x = \sim - x$.

Theorem

If $\mathbf{A}$ *is a PE/PO-algebra, effect algebra or orthoalgebra, then* $\bar{\mathbf{A}}$ *is an involutive residuated poset.*

Some References

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