

The continuous weak (Bruhat) order and mix $*$ -quantales

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Plan

Permutations, words, and paths

The continuous order in dimension 2:
the mix \star -quantale $Q_V(\mathbb{I})$

The continuous order, dimension > 2

Conclusions

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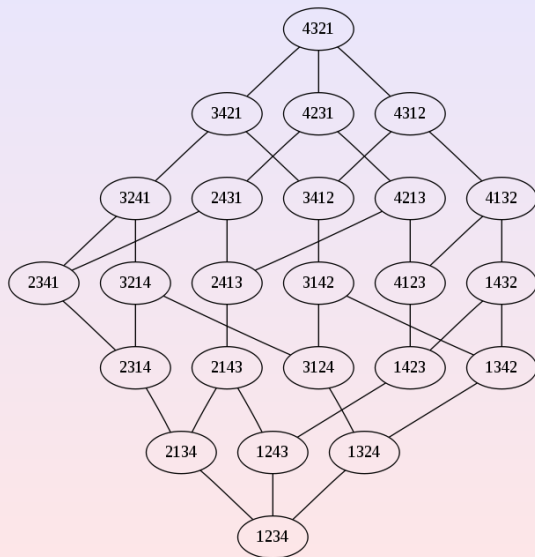
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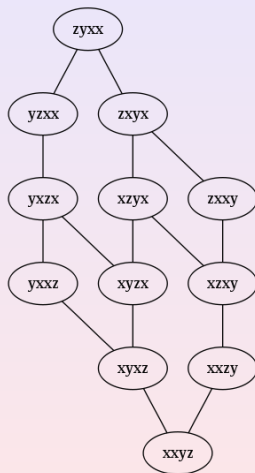
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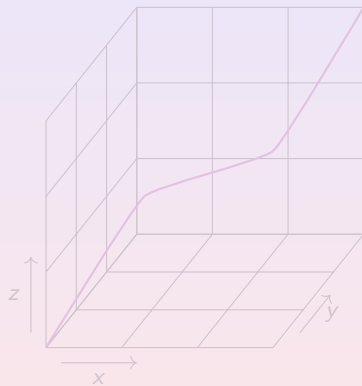
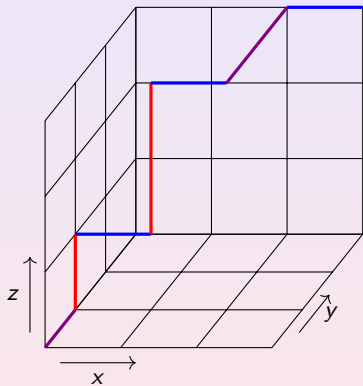
The weak Bruhat order, aka the permutohedra $P(n)$



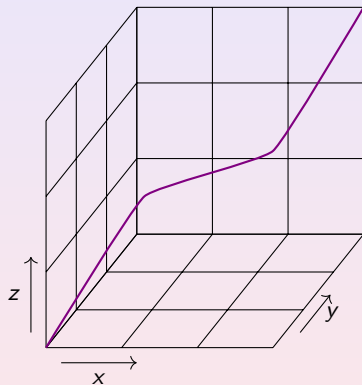
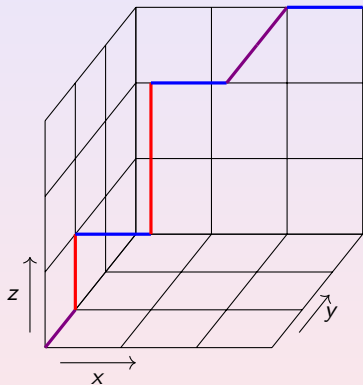
Multinomial lattices



Are there continuous multinomial lattices?



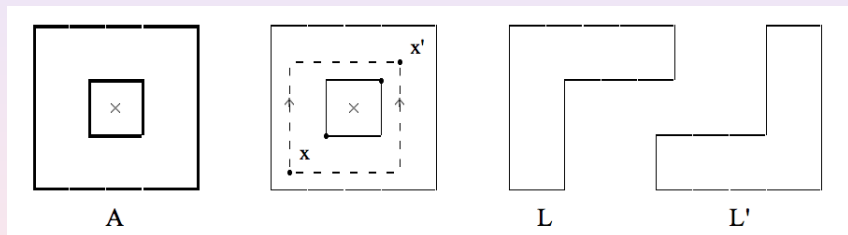
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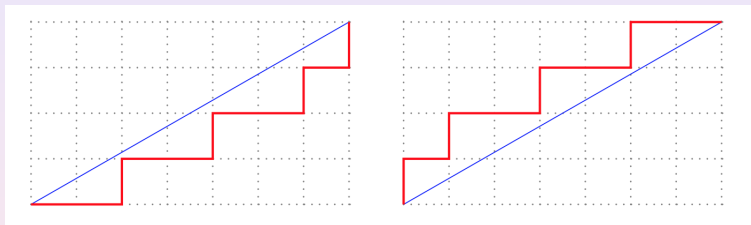
Pre-motivations

- Understand multinomial lattices via geometry.
- Lifting some obvious facts/remarks concerning
 - directed homotopy,
 - discrete geometry,from dimension 2 to higher dimensions.

Directed homotopy and lattice congruences

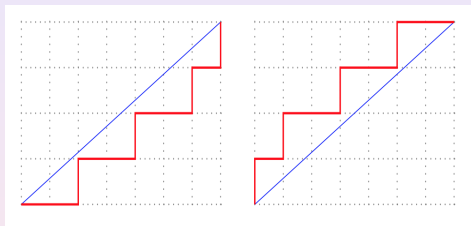


Discrete geometry and Christoffel words



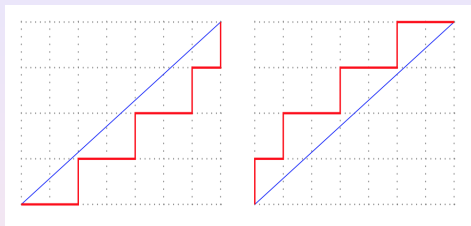
Christoffel words are images of the diagonal via right/left adjoints:

Discrete geometry and Christoffel words



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Christoffel words are images of the diagonal via right/left adjoints:

$$\begin{array}{ccc}
 & \xleftarrow{\ell} & \\
 & \xleftarrow{\iota} & \xrightarrow{\iota} \\
 & \xleftarrow{\rho} &
 \end{array}
 \begin{array}{c}
 L(7, 4) \\
 \\
 Q_V(\mathbb{I})
 \end{array}$$

Yes, there are continuous multinomial lattices!

From TACL 2011:

Theorem

For each $d \geq 3$, there is a lattice structure on the set of images of continuous paths from \mathbb{I} to \mathbb{I}^d .

The focus here:

- How the algebraic structure of \star -quantale (involutive residuated lattice) contributes to this result.

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The lattice $Q_V(\mathbb{I})$

Let, from now on, $\mathbb{I} := [0, 1]$.

Proposition

The following sets are (equal or) in bijective correspondence:

- $\{ C \subseteq \mathbb{I}^2 \mid C \text{ image of a monotone continuous path } \pi : \mathbb{I} \rightarrow \mathbb{I}^2 \text{ s.t. } \pi(0) = \vec{0} \text{ and } \pi(1) = \vec{1} \},$
- $\{ C \subseteq \mathbb{I}^2 \mid C \text{ chain, dense, complete} \},$
- $\{ C \subseteq \mathbb{I}^2 \mid C \text{ maximal chain of } \mathbb{I}^2 \},$
- $\{ f : \mathbb{I} \rightarrow \mathbb{I} \mid f \text{ is join-continuous} \},$
- $\{ f : \mathbb{I} \rightarrow \mathbb{I} \mid f \text{ is meet-continuous} \}.$

The lattice $Q_V(\mathbb{I})$

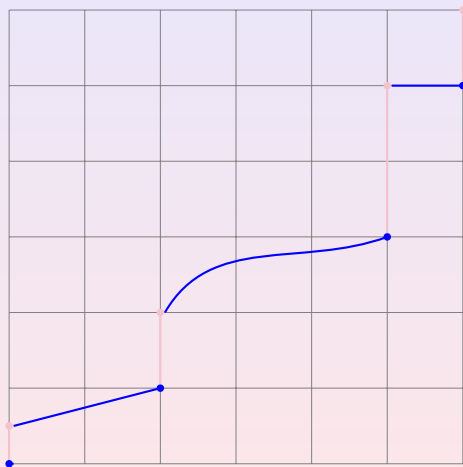
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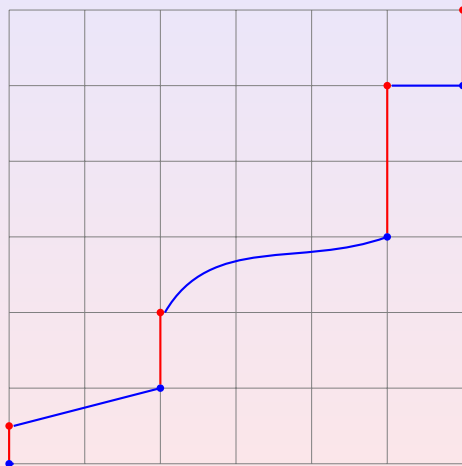
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From join-continuous functions to maximal chains



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Few properties of $Q_{\vee}(\mathbb{I})$

Let $Q_{\vee}(\mathbb{I})$ be the set of join-continuous functions from \mathbb{I} to \mathbb{I} .

The order on $Q_{\vee}(\mathbb{I})$ is pointwise.

Proposition

- $Q_{\vee}(\mathbb{I})$ is a distributive complete lattice,
- every $f \in Q_{\vee}(\mathbb{I})$ is a \bigwedge and a \bigvee of some step function (with a finite no. of steps *with rational steps*),

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More properties of $Q_V(\mathbb{I})$

- It is (canonically) a quantale:

$$f \otimes g := g \circ f, \quad 1 := id.$$

- It is a non-commutative \star -(autonomous) quantale, aka an involutive residuated lattice.

That is, it comes with an (antitone) involution $(-)^*$ s.t., defining

$$f \oplus g := (g^* \otimes f^*)^*$$

we have

$$f \otimes g \leq h \text{ iff } f \leq h \otimes g^* \text{ iff } g \leq f^* \otimes h.$$

- It is mix:

$$f \otimes g \leq f \oplus g.$$

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- Let $Q_{\wedge}(\mathbb{I})$ be the set of meet-continuous functions from \mathbb{I} to itself.
- Put:

$$f^{\wedge}(x) := \bigwedge_{x < y} f(y), \quad g^{\vee}(x) := \bigvee_{z < x} g(z).$$

Then $Q_{\vee}(\mathbb{I})$ and $Q_{\wedge}(\mathbb{I})$ are (covariantly) isomorphic posets.

- We have then

$$f^{\star} := (\text{right-adj}(f))^{\vee} = \text{left-adj}((f)^{\wedge}),$$

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Skew enrichments/metrics on a \star -quantale

A skew metric (enrichment) on a \star -quantale Q is a pair (X, δ) such that, for each $i, j \in X$ with $i \neq j$,

$$\begin{aligned}\delta(i, k) &\leq \delta(i, j) \oplus \delta(j, k), \\ \delta(i, j) &= \delta(j, i)^*.\end{aligned}$$

If $1 = 0$, you can also ask

$$\delta(i, i) = 0.$$

Clopens as skew enrichments

Let $[d] := \{1, \dots, d\}$ and $[d]_2 := \{(i, j) \mid 1 \leq i < j \leq d\}$.

For $f \in Q^{[d]_2}$, we say that f is

- *closed* if

$$f_{i,j} \otimes f_{j,k} \leq f_{i,k},$$

for each $i < j < k$,

- *open* if

$$f_{i,k} \leq f_{i,j} \oplus f_{j,k},$$

for each $i < j < k$,

- *clopen* if it is both closed and open.

Lemma

There is a bijection between skew enrichments on the set $[d]$ and clopen sets of the poset $Q^{[d]_2}$.

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Theorem

- For each $d \geq 2$ and each mix \star -quantale Q , the set $L_d(Q)$ of clopen tuples of $Q^{[d]^2}$ is, with the coordinatewise ordering, a lattice.
- The construction $Q \mapsto L_d(Q)$ is a limit preserving functor from the category of mix ℓ -bisemigroups to the category of bounded lattices.

Roughly speaking, an ℓ -bisemigroup is the $\otimes, \oplus, \perp, \vee, \top, \wedge$ -reduct of a \star -quantale.

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Examples

- If $Q = 2$, then clopen tuples are in bijection with transitive cotransitive subsets of $[d]_2$; these are in bijection with permutations of $[d]$.
 $L_d(2)$ is the weak Bruhat ordering.
- If Q is the Sugihara monoid on the chain 3, then clopen tuples and their ordering correspond to pseudo-permutations [Krob et al. 2000].
- If $Q = Q_V(\{0, \dots, n\})$, then elements of $L_d(Q)$ are in bijection with maximal chains in the cube $\{0, 1, \dots, n\}^d$, i.e. words $w \in [d]^*$ such that $|w|_i = n, i = 1, \dots, d$.
 $L_d(Q)$ is the multinomial lattice $L(\underbrace{n, \dots, n}_{d \text{ times}})$.

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When Q is $Q_V(\mathbb{I})$

Theorem

Let $d \geq 3$. The following sets are equal or in bijection:

- $\{ C \subseteq \mathbb{I}^d \mid C \text{ is a maximal chain} \}$
- $\{ C \subseteq \mathbb{I}^d \mid C \text{ chain, dense, complete} \}$
- $\{ \text{images of continuous monotone paths } \pi : \mathbb{I} \longrightarrow \mathbb{I}^d$
 $\text{s.t. } \pi(0) = \vec{0} \text{ and } \pi(1) = \vec{1} \}$
- $\{ f \in Q_V(\mathbb{I})^{[d]^2} \mid f \text{ is clopen} \}$
- $L_d(Q_V(\mathbb{I}))$.

Corollary

The set of maximal chains of \mathbb{I}^d is a lattice, with the ordering given projection-wise.

Structural properties of $L_d(Q_V(\mathbb{I}))$, $d \geq 3$

- It is not distributive.
- $L_d(Q_V(\mathbb{I}))$ has no completely join-irreducible elements nor compact elements.
- Every $f \in L_d(Q_V(\mathbb{I}))$ is a \bigvee and a \bigwedge of join-irreducible elements.
- Join-irreducible elements can be identified with points in \mathbb{I}^d .
- Not every $f \in L_d(Q_V(\mathbb{I}))$ is a \bigvee and a \bigwedge of join-irreducible elements with rational coordinates.
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Rephrasing the previous observations

- A bound-preserving embedding $\{0, \dots, n\} \rightarrow \mathbb{I}$ firstly yields an ℓ -bisemigroup embedding

$$Q_V(\{0, \dots, n\}) \rightarrow Q_V(\mathbb{I})$$

and then a lattice embedding

$$L_d(Q_V(\{0, \dots, n\})) \rightarrow L_d(Q_V(\mathbb{I})).$$

- According to the previous statement, $L_d(Q_V(\mathbb{I}))$ is the Dedekind-MacNeille completion of the colimit of these embeddings.
- If we restrict to the embeddings of the form

$$i \in \{0, \dots, n\} \mapsto \frac{i}{n} \in \mathbb{I}$$

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$$L_d(Q_V(\{0, \dots, n\})) \rightarrow L_d(Q_V(\mathbb{I})).$$

- According to the previous statement, $L_d(Q_V(\mathbb{I}))$ is the Dedekind-MacNeille completion of the colimit of these embeddings.
- If we restrict to the embeddings of the form

$$i \in \{0, \dots, n\} \mapsto \frac{i}{n} \in \mathbb{I}$$

then $L_d(Q_V(\mathbb{I}))$ is not anymore the Dedekind-MacNeille completion of the respective colimit: we need two steps to complete all.

Perfect chains and functoriality of $Q_V(-)$

- A chain I is perfect if it is complete and the maps $(-)^{\wedge}$ and $(-)^{\vee}$ defined by

$$f^{\wedge}(x) := \bigwedge_{x < y} f(y), \quad g^{\vee}(x) := \bigvee_{z < x} g(z),$$

are inverse isomorphisms between $Q_V(I)$ and $Q_{\wedge}(I)$.

- If I is perfect, then $Q_V(I)$ is a mix \star -quantale.
- \mathbb{I} is perfect, as well as any finite chain $\{0, \dots, n\}$.
- If $\iota : I_0 \rightarrow I_1$ is a complete (preserves arbitrary \vee and \wedge) embedding between perfect chains, then we can “right-Kan extend” $f \in Q_V(I_0)$ to $Q_V(I_1)$.
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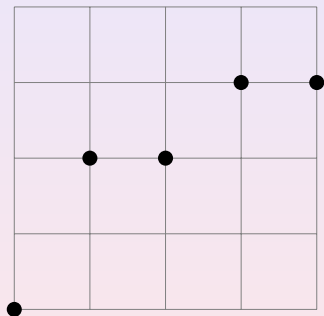
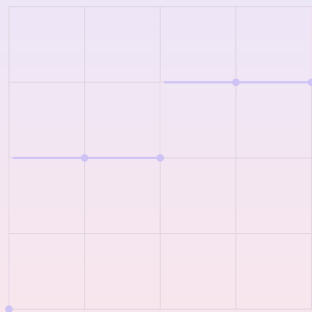
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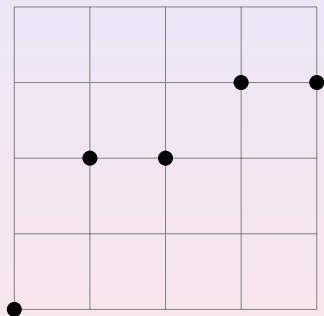
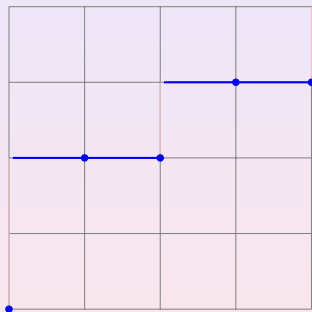
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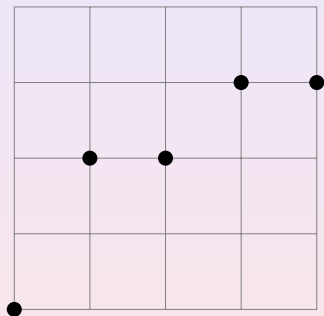
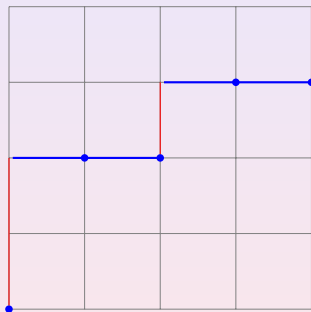
Right Kan extending as drawing paths

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Plan

Permutations, words, and paths

The continuous order in dimension 2:

the mix \star -quantale $Q_V(\mathbb{I})$

The continuous order, dimension > 2

Conclusions

Post-motivations, challenges

Tackle pre-motivations ...

Decidability:

- Lattice theoretic:
 - Equational theory of $L_d(Q_V(\mathbb{I}))$, $d \geq 3$?
 - For fixed d , is the equational theory of the $L_d(Q_V(\mathbb{I}))$ the same as the theory of the multinomial lattices of dimension d ?
- Equational theory of $Q_V(\mathbb{I})$, as an *involutive residuated lattice*?

Algebra:

- Understand structural properties of $L_d(Q)$ in terms of the abstract properties of a quantale Q .
- How many lattices arise as $L_d(Q)$ for some mix \star -quantale Q ?
- Discover new lattices from mix \star -quantales (ongoing work with all the Sugihara monoids).

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Post-motivations, challenges II

Enumerative combinatorics:

- We have

$$\begin{aligned}L(n, n) &= Q_V(\{0, 1, \dots, n\}) \\ &= \text{quantale of weaking-relations/distributors/bimodules} \\ &\quad \text{from the chain } \{1, \dots, n\} \text{ to itself.}\end{aligned}$$

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Thanks for your attention