T-Norm Based Operations in Arrow Categories

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- Categories of L-Fuzzy Relations.
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Classical, Fuzzy and L-Fuzzy Relations

Classical relation: Boolean values $R : A \times B \to \mathbb{B}$

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

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Fuzzy relation:	$[0,1] \subseteq \mathbb{R}$	$R : A \times B \to [0,1]$	$\begin{pmatrix} 0.1 & 0.8 & 0.0 \\ 1.0 & 0.4 & 0.9 \\ 0.0 & 0.2 & 0.1 \end{pmatrix}$

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Fuzzy relation: $[0,1] \subseteq \mathbb{R}$ $R : A \times B \to [0,1]$

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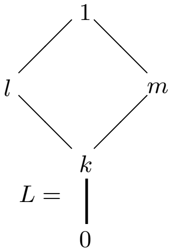
L-fuzzy relation: Lattice L $R : A \times B \to L$

$$\begin{pmatrix} 1 & k & l \\ 0 & k & m \\ 0 & 1 & l \end{pmatrix}$$



Operations on Relations

Let $(L, \wedge, \vee, 0, 1, \rightarrow)$ be a complete Heyting algebra, $Q, R : A \rightarrow B$, $S : B \rightarrow C$, and $T : D \rightarrow C$ be L-fuzzy relations, and $x, x' \in A$, $y \in B$, $z \in C$, and $u \in D$:

$$\perp\!\!\!\perp_{AB}(x,y) := 0, \qquad \top\!\!\!\top_{AB}(x,y) := 1, \qquad \mathbb{I}_A(x,x') := \begin{cases} 1 & \text{iff } x = y \\ 0 & \text{otherwise} \end{cases}$$

$$(Q \sqcap R)(x,y) := Q(x,y) \wedge R(x,y), \qquad (Q \sqcup R)(x,y) := Q(x,y) \vee R(x,y)$$

$$(Q;S)(x,z) := \bigvee_{y \in B} Q(x,y) \wedge S(y,z), \qquad (S/T)(u,y) := \bigwedge_{z \in C} T(u,z) \rightarrow S(y,z)$$

$$Q^{\smile}(y,x) := Q(x,y)$$

Dedekind Categories

Definition

A Dedekind category $\mathcal{R}$ is a category satisfying the following:

1. For all objects A and B the collection $\mathcal{R}[A,B]$ is a complete distributive lattice with operations $\sqcap, \sqcup, \sqsubseteq, \perp\!\!\!\perp_{AB}, \top\!\!\!\top_{AB}$.
2. There is a monotone operation $\breve{}$ (called conversion) so that for all relations $Q : A \to B$ and $R : B \to C$:
$$(Q;R)^\smile = R^\smile;Q^\smile, \qquad (Q^\smile)^\smile = Q.$$
3. For all relations $Q : A \to B, R : B \to C$ and $S : A \to C$ the modular law holds:
$$Q;R \sqcap S \sqsubseteq Q;(R \sqcap Q^\smile;S).$$
4. For all relations $R : B \to C$ and $S : A \to C$ there is a relation $S/R : A \to B$ (called the left residual of S and R) so that for all $Q : A \to B$ the following holds:
$$Q;R \sqsubseteq S \iff Q \sqsubseteq S/R.$$

Special L-fuzzy Relations I

Relations corresponding to elements in L:

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Special L-fuzzy Relations I

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scalar relations

Definition

1. A relation $J : A \to B$ is called an ideal iff $\top\!\!\top_{AA};J;\top\!\!\top_{BB} = J$.
2. A relation $\alpha : A \to A$ is called a scalar on A iff $\alpha \sqsubseteq \mathbb{I}_A$ and $\top\!\!\top_{AA};\alpha = \alpha;\top\!\!\top_{AA}$.

Special L-fuzzy Relations II

Crisp relation:

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

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BUT...

Theorem

There is no formula φ in the language of Dedekind categories such that for all lattices L and L-relations $R : A \to B$ we have

$$Rel(L) \models \varphi(R) \quad \iff \quad R \text{ is crisp.}$$

Arrow Operations

$\therefore$ We need more operations to capture the notion of cripsness.

Let $Q : A \to B$ be a L-fuzzy relation, $x \in A$, and $y \in B$:

$$Q^{\downarrow}(x,y) := \begin{cases} 1 & \text{iff } Q(x,y) = 1 \\ 0 & \text{otherwise} \end{cases}, \qquad Q^{\uparrow}(x,y) := \begin{cases} 1 & \text{iff } Q(x,y) \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

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Obviously, a relation Q is crisp iff $Q^{\downarrow} = Q$ or equivalently $Q^{\uparrow} = Q$.

Arrow Categories I

Definition

An arrow category $\mathcal{A}$ is a non-trivial Dedekind category ($\top\!\!\top_{AB} \neq \perp\!\!\perp_{AB}$ for all objects A and B) together with two operations $^{\uparrow}$ and $^{\downarrow}$ satisfying the following:

(1) $R^{\uparrow}, R^{\downarrow} : A \to B$ for all $R : A \to B$.

(2) $(^{\uparrow}, ^{\downarrow})$ is a Galois correspondence, i.e., $Q^{\uparrow} \sqsubseteq R$ iff $Q \sqsubseteq R^{\downarrow}$.

(3) $(R^{\smile}; S^{\downarrow})^{\uparrow} = R^{\uparrow\smile}; S^{\downarrow}$ for all $R : B \to A$ and $S : B \to C$.

(4) $(Q \sqcap R^{\downarrow})^{\uparrow} = Q^{\uparrow} \sqcap R^{\downarrow}$ for all $Q, R : A \to B$.

(5) If $\alpha_A \neq \perp\!\!\perp_{AA}$ is a non-zero scalar, then $\alpha_A^{\uparrow} = \mathbb{I}_A$.

Arrow Categories II

Theorem

Let $\mathcal{A}$ be an arrow category. Then we have:

1. $\mathcal{A}$ *is uniform, i.e., we have* $\top\!\!\top_{AB};\top\!\!\top_{BC} = \top\!\!\top_{AC}$ *for all objects* A, B *and* C.

Arrow Categories II

Theorem

Let $\mathcal{A}$ be an arrow category. Then we have:

1. *$\mathcal{A}$ is uniform, i.e., we have $\top\!\!\top_{AB};\top\!\!\top_{BC} = \top\!\!\top_{AC}$ for all objects A, B and C.*
2. *The complete Heyting algebras $\mathrm{Sc}(A)$ of scalar relations on two objects are isomorphic.*

T-Norm based Operations

If L is a complete lattice, then $\langle L, * \rangle$ with a binary operation $*$ on L is called a partially ordered (Abelian) monoid iff

1. $\langle L, *, 1 \rangle$ is a bounded Abelian monoid, i.e., $*$ is associative and commutative with the greatest element 1 of L as neutral element,
2. $*$ is monotonic in both parameters.

If $*$ distributes over arbitrary unions, then we call $*$ continuous.

- We call the operation of a continuous partially ordered Abelian monoid a t-norm for short.
- Please note that a t-norm $*$ also provides a residual, i.e., there is a binary operation $\twoheadrightarrow$ such that $x * y \sqsubseteq z$ iff $x \sqsubseteq y \twoheadrightarrow z$ for all $x, y, z \in L$.

Operations on Relations induced by a t-Norm

In the case of L-relations and a given t-norm $*$ on L we may define two new operations on relations by

- $(Q * R)(x,y) := Q(x,y) * R(x,y)$,
- $(Q \mathbin{*\!\!;} S)(x,z) := \bigsqcup\limits_{y \in B} Q(x,y) * S(y,z)$.

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In the case of L-relations and a given t-norm $*$ on L we may define two new operations on relations by

- $(Q * R)(x, y) := Q(x, y) * R(x, y)$,
- $(Q \underset{;}{*} S)(x, z) := \bigsqcup_{y \in B} Q(x, y) * S(y, z)$.

In the case of an arrow category satisfying the α-cut theorem, i.e.,

$$\bigsqcup_{\alpha \text{ scalar}} \alpha_A; (\alpha_A \backslash R)^{\downarrow} = R$$

for all R, and a given t-norm on $\mathrm{Sc}(A)$ we may define the operations above abstractly by

- $Q * R := \bigsqcup_{\alpha, \beta \text{ scalars}} (\alpha_A * \beta_A); ((\alpha_A \backslash Q)^{\downarrow} \sqcap (\beta_A \backslash R)^{\downarrow})$,
- $Q \underset{;}{*} R := \bigsqcup_{\alpha, \beta \text{ scalars}} (\alpha_A * \beta_A); (\alpha_A \backslash Q)^{\downarrow}; (\beta_B \backslash S)^{\downarrow}$.

$*$-based Meet I

Lemma

Suppose $Q, R : A \to B$ are L-relations. Then we have

1. *(T1) $*$ is associative and commutative,*
2. *(T2) $*$ is continuous, and, hence, also monotonic,*
3. *(T3) $(Q * R)^{\smile} = Q^{\smile} * R^{\smile}$,*
4. *(T4) $Q * R^{\downarrow} = Q \sqcap R^{\downarrow}$.*

$*$-based Meet II

Theorem

Let $\mathcal{R}$ be an arrow category and $\mathrm{Sc}[A]$ the set of scalars on A. Then the partially ordered monoids $\langle \mathrm{Sc}[A], * \rangle$ are isomorphic.

$*$-based Meet II

Theorem

Let $\mathcal{R}$ be an arrow category and $\mathrm{Sc}[A]$ *the set of scalars on* A*. Then the partially ordered monoids* $\langle \mathrm{Sc}[A], * \rangle$ *are isomorphic.*

Theorem

Let $\mathcal{R}$ be an arrow category satisfying the α-cut theorem. If $*$ satisfies (T1)-(T4), then we have

$$Q * R = \bigsqcup_{\alpha,\beta \ scalars} (\alpha * \beta);((\alpha \backslash Q)^{\downarrow} \sqcap (\beta \backslash R)^{\downarrow}).$$

$*$-based Composition I

Lemma

Suppose $P, Q : A \to B$ and $R, S : B \to C$ are L-relations. Then we have

1. *(CT1)* $\overset{*}{;}$ *is associative,*
2. *(CT2)* $\overset{*}{;}$ *is continuous, and, hence, also monotonic,*
3. *(CT3)* $(Q \overset{*}{;} R)^{\smile} = R^{\smile} \overset{*}{;} Q^{\smile}$,
4. *(CT4)* $Q \overset{*}{;} R^{\downarrow} = Q; R^{\downarrow}$,
5. *(CT5)* $(P * Q) \overset{*}{;} (R * S) \sqsubseteq (P \overset{*}{;} R) * (Q \overset{*}{;} S)$.

$*$-based Composition II

The following versions of the modular inclusion $Q; R \sqcap S \sqsubseteq Q; (R \sqcap Q^{\smile}; S)$ are valid if the regular operations are replaced by the $*$-based versions:

Lemma

For L-relations $P, Q : A \to B, R : B \to C$ and $S : A \to C$ we have

1. *(CT6)* $Q; R * S \sqsubseteq Q; (R \sqcap Q^{\smile} \overset{*}{;} S)$,
2. *(CT7)* $Q \overset{*}{;} R * S \sqsubseteq Q; (R * Q^{\smile} \overset{*}{;} S)$,

$*$-based Composition II

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Lemma

For L-relations $P, Q : A \to B, R : B \to C$ and $S : A \to C$ we have

1. *(CT6)* $Q;R * S \sqsubseteq Q;(R \sqcap Q^{\smile} \stackrel{*}{;} S)$,
2. *(CT7)* $Q \stackrel{*}{;} R * S \sqsubseteq Q;(R * Q^{\smile} \stackrel{*}{;} S)$,
3. *(CT8)* $(P * Q) \stackrel{*}{;} R * S \sqsubseteq P \stackrel{*}{;} (R * Q^{\smile} \stackrel{*}{;} S)$.

$*$-based Composition III

All other potential replacements lead to invalid inclusions.

$\sqsupseteq$	$Q;R\sqcap S$	$Q;R*S$	$Q\overset{*}{;}R\sqcap S$	$Q\overset{*}{;}R*S$
$Q;(R\sqcap Q^{\smile};S)$	$+$	$+$	$+$	$+$
$Q;(R\sqcap Q^{\smile}\overset{*}{;}S)$	$-$	$+$	$-$	$+$
$Q;(R*Q^{\smile};S)$	$-$	$-$	$-$	$+$
$Q;(R*Q^{\smile}\overset{*}{;}S)$	$-$	$-$	$-$	$+$
$Q\overset{*}{;}(R\sqcap Q^{\smile};S)$	$-$	$-$	$-$	$-$
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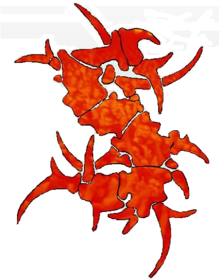
$*$-based Composition IV

Theorem

Let $\mathcal{R}$ be an arrow category satisfying the α-cut theorem. If $\overset{*}{;}$ satisfies (CT1)-(CT8), then we have

$$Q\overset{*}{;}R = \bigsqcup_{\alpha,\beta\ scalars} (\alpha_A * \beta_A);(\alpha_A\backslash Q)^{\downarrow};(\beta_B\backslash R)^{\downarrow}.$$



Thank you
for your attention!

Questions?