On the computational complexity of non-dictatorial aggregation

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Outline

1. Introduction: Preference Aggregation
2. Preliminaries
   - Framework
   - Statement of problems
   - Characterizations (previous results)
3. Algorithms (new results)
   - Universal Algebra
   - Possibility Domains
   - Uniform Possibility Domains

Preference Aggregation

- $\mathcal{O} = \{o_1, \ldots, o_k\}$, $k \geq 3$.
- $n \geq 2$ individuals,
- $<_i$: ranking of i-th agent [strict total ordering of $\mathcal{O}$].
- X set of strict total orderings of $\mathcal{O}$.

▶ Social Welfare Function $SWF : X^n \mapsto X$.

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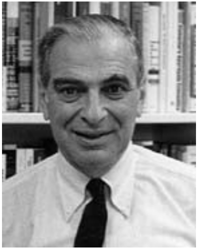
▶ Social Welfare Function $SWF : X^n \mapsto X$.

▶ J. K. Arrow (1951): Showed that non-trivial preference aggregation is impossible given only some mild axioms that a SWF should satisfy.

John Kenneth Arrow, 1921-2017

The conditions are:

1. Universality (unrestricted domain):
   $SWF : X^n \mapsto X$,
2. Independency (independency of irrelevant alternatives): ranking o_i, o_j does not depend on the ranking of any other o_l.
3. Pareto efficiency (unanimity): if everyone ranks o_i higher than o_j, o_i should be ranked higher by the SWF.



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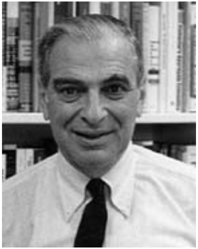

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And then we have:

Theorem (General Possibility, 1951)

Any SWF satisfying 1, 2, 3 is a dictatorship: the outcome is the ranking of a single individual.

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- n voters, m issues.
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- $X \subseteq \prod_{j=1}^{m} A_j$ set of feasible voting patterns.
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Non-degeneracy conditions

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1. Universality and IIA are "built in".
2. Conservativeness (supportiveness):

$$\text{if } x_j = (x_j^1, ..., x_j^n) \in A_j^n \text{ then } f_j(x_j) \in \{x_j^1, \ldots, x_j^n\},$$

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   for all $j = 1, \dots, m$.

   - Stronger notion than unanimity (equivalent in the Boolean framework).
   - The social outcome for each issue must be equal to at least one individual's position.

Aggregators cont.

$$\bar{f} \begin{pmatrix} x_1^1 & \cdots & x_j^1 & \cdots & x_m^1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_1^i & \cdots & x_j^i & \cdots & x_m^i \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_1^n & \cdots & x_j^n & \cdots & x_m^n \end{pmatrix}$$

Aggregators cont.

$$\begin{pmatrix} \color{red}{f_1} & & & & \\ \color{red}{\frown} & & & & \\ x_1^1 & \cdots & x_j^1 & \cdots & x_m^1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_1^i & \cdots & x_j^i & \cdots & x_m^i \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_1^n & \cdots & x_j^n & \cdots & x_m^n \\ \color{red}{\smile} & & & & \\ \shortparallel & & & & \\ f_1(x_1) & & & & \end{pmatrix}$$

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$$= \bar{f}(x^1, \ldots, x^n) \in X.$$

Majority/minority aggregators

- $\bar{f} = (f_1, \ldots, f_m)$ is a majority/minority aggregator if for all j, $f_j(x, x, y) = f_j(x, y, x) = f_j(y, x, x) = x \mid y$.
- X admits a majority aggregator if and only if it admits a ternary $\bar{f} = (f_1, \ldots, f_m)$ such that, for all j, B_j (binary) $f_j \restriction_{B_j} = maj : \{0,1\}^3 \mapsto \{0,1\}$ s.t.

$$\mathrm{maj}(x, y, z) = \begin{cases} x & \text{if } x = y \text{ or } x = z, \\ y & \text{if } y = z. \end{cases}$$

- Accordingly, it admits a minority aggregetor if and only if it admits a ternary $\bar{f} = (f_1, \ldots, f_m)$ such that, for all j, B_j (binary) $f_j \restriction_{B_j} = \oplus : \{0,1\}^3 \mapsto \{0,1\}$, where $\oplus(x, y, z) = x \oplus y \oplus z$ is the binary addition modulo 2.

Possibility Domains

Definition

An n-ary aggregator $\bar{f}$ is called dictatorial for X if there is a $d \in \{1, \ldots, n\}$ such that: $(f_1, \ldots, f_m) = (pr_d^n, \ldots, pr_d^n)$.

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Example (non-dictatorial aggregators): Majority and minority aggregators, binary $\bar{f} = (f_1, \ldots, f_m)$ s.t. there exists a binary $B_j \subseteq A_j$: $f_j \restriction_{B_j} = \wedge, \vee$.

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Uniform Possibility Domains

Definition (Kirousis, Kolaitis, Livieratos 2017)

$\bar{f}$ is uniform non-dictatorial if for all j, B_j (binary) $f_j \restriction_{B_j}$ is not a projection.

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Possibility domains

Can we "efficiently" decide if a domain X is a possibility domain?

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- Is there an algorithm that on input X, outputs the answer to the above questions, in polynomial time to the number of issues m and the size $|X|$ of X?
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An aggregator of any arity n will do! The search space it too big!

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Avoiding dictators

Theorem (KKL, 2017)

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- *a majority aggregator or*
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Theorem (KKL, 2017)

The following are equivalent:

- *X is a uniform possibility domain.*
- *For every $j, B_j \subseteq A_j$ (binary), either there is binary aggregator $\bar{f}$ s.t. $f_j{\restriction}_{B_j} \in \{\wedge, \vee\}$ or a ternary $\bar{f}$ s.t. $f_j{\restriction}_{B_j} \in \{\mathrm{maj}, \oplus\}$.*

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Polymorphisms

- A finite set.
- $R \subseteq A^m$ m-ary relation.
- $f : A^n \mapsto A$ n-ary polymorphism of R if:

$$x^1, \ldots, x^n \in R \Rightarrow f(x^1, \ldots, x^n)$$
$$:= (f(x_1^1, \ldots, x_1^n), \ldots, f(x_m^1, \ldots, x_m^n)) \in R.$$

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▶ We will assume all polymorphisms are conservative/supportive (their output is part of the input).

Polymorphisms as aggregators

In order to use polymorphisms in our framework:

- Mark the elements of each A_j in a different way (color, number, etc.).
- Pass from X to $\tilde{X} \subseteq \bigcup_{j=1}^{m} A_j$.

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- Pass from X to $\tilde{X} \subseteq \bigcup_{j=1}^{m} A_j$.
- ★ Any aggregator for X can be translated to a polymorphism of $\tilde{X}$ and vice versa.

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Theorem 1

Theorem (KKL, 2018)

There is a polynomial-time algorithm that decides if a domain X is a possibility domain, and if it is, that produces a non-dictatorial aggregator of arity at most three.

★ We assume that X is extensively part of the input.

Proof Outline

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2. For the binary aggregators, define H_X with vertices uu'_j, where $u, u' \in X_j$ s.t. $u \neq u'$ and edges $uu'_k \to vv'_l$, where:
   - $k \neq l$,
   - there are $z, z' \in X$ extending $(u, v), (u', v')$ and
   - there is no $w \in X$ extending (u, v') s.t. $w_j \in \{z_j, z'_j\}$.

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1. There are known algorithms to check if $\tilde{X}$ has a majority or a minority polymorphism (Bressiére & Carbonnel 2013, 2016).
2. For the binary aggregators, define H_X with vertices uu'_j, where $u, u' \in X_j$ s.t. $u \neq u'$ and edges $uu'_k \to vv'_l$, where:
   - $k \neq l$,
   - there are $z, z' \in X$ extending $(u, v), (u', v')$ and
   - there is no $w \in X$ extending (u, v') s.t. $w_j \in \{z_j, z'_j\}$.

★ Intuitively, $uu'_k \to vv'_l$ means that, for any binary aggregator $\bar{f}$: if $f_k(u, u') = u$, then $f_l(v, v') = v$.

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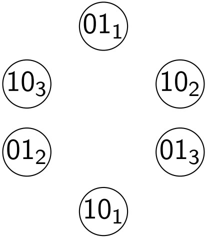
3. X has a binary non-dictatorial aggregator if and only if H_X is not strongly connected.

★ H_X can be constructed in polynomial time and there are known polynomial algorithms (e.g. Kosaraju's and Tarjan's algorithms), that check the strong connectedness of a graph.

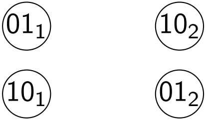
H_X

H_{X_1}



$X_1 = \{(0,0,1),(0,1,0),(1,0,0)\}$

H_{X_2}



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H_X

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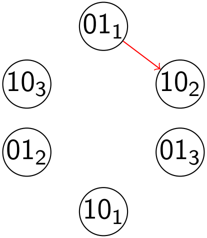
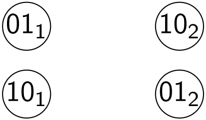


H_{X_2}



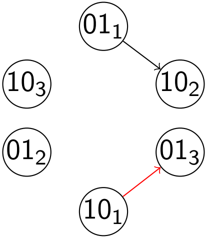
$X_2 = \{(0,1),(1,0)\}$

$X_1 = \{(0,0,1),(0,1,0),(1,0,0)\}$

$(0,1,0),(1,0,0) \in X_1$, $(0,0,0) \notin X_1$

H_X

H_{X_1}



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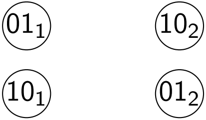
$(0,0,1),(1,0,0) \in X_1,$

$(1,*,1) \notin X_1$

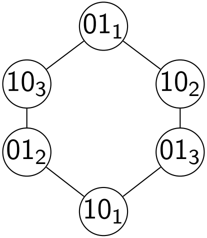
H_{X_2}



$X_2 = \{(0,1),(1,0)\}$

H_X

H_{X_1}



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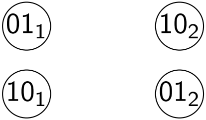
H_{X_2}



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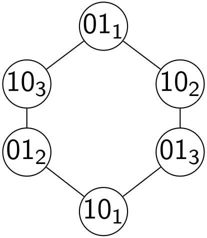
H_X

H_{X_1}



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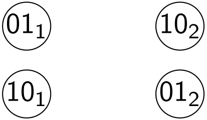

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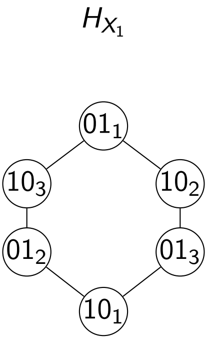
H_X



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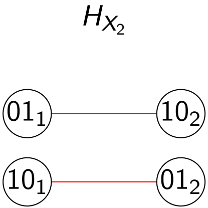

$(0,1),(1,0) \in X_2$,
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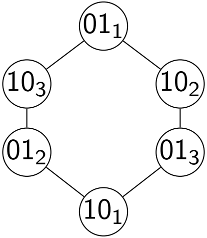
H_X

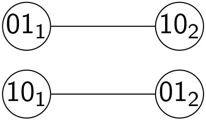
H_{X_1}



$X_1 = \{(0,0,1),(0,1,0),(1,0,0)\}$

Strongly Connected

H_{X_2}



$(0,1),(1,0) \in X_2$,
$(0,0),(1,1) \notin X_2$.

Not Strongly connected

Outline

Theorem 2

Theorem (KKL, 2018)

There is a polynomial-time algorithm that decides if a domain X is a uniform possibility domain, and if it is, that produces a uniform non-dictatorial aggregator of arity at most three.

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There is a polynomial-time algorithm that decides if a domain X is a uniform possibility domain, and if it is, that produces a uniform non-dictatorial aggregator of arity at most three.

Proof:

- (Carbonnel, 2016): There is a polynomial-time algorithm that decides if for all binary $B \subseteq A$ there is a polymorphism f of a constraint language Γ (set of relations over A) such that f is either binary and $f\restriction_B \in \{\wedge, \vee\}$, or f is ternary and $f\restriction_B \in \{maj, \oplus\}$ and that, if there is, produces one.
- This translates to the aggregator of the characterization theorem for uniform possibility domains!

Future Work

Open problem: What happens when X is given implicitly?

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- Integrity constraint (Grandi, Endriss 2010): $X_\phi \subseteq \{0,1\}^m$: set of satisfying assignments of a propositional formula ϕ.

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- Integrity constraint (Grandi, Endriss 2010): $X_\phi \subseteq \{0,1\}^m$: set of satisfying assignments of a propositional formula ϕ.
- Agenda: $\bar{\phi} = (\phi_1, \ldots, \phi_m)$, where each individual decides if each ϕ_j is true or false and $X_{\bar{\phi}}$ contains all consistent such judgments.

Thank you!