A Modal and Relevance Logic for Qualitative Spatial Reasoning

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Content

- Introduction/Motivation.
- Modal and Relevance Logic.
- The Logic.
- Natural Deduction.

Motivation

- Logic for reasoning about regions, i.e., regular closed sets.

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- In previous logics models are topological spaces and regions are certain predicates.
- We want that the models of the logic are Boolean Contact Algebras (BCAs) so that elements are regions.

Boolean Contact Algebras

Definition

A binary relation C on a BA $\mathcal{B}$ is called contact relation if it satisfies the following axioms for all $x, y, z \in B$:

(C_0)	Null disconnectedness	$xCy \Rightarrow x, y \neq 0$
(C_1)	Reflexivity	$x \neq 0 \Rightarrow xCx$
(C_2)	Symmetry	$xCy \Leftrightarrow yCx$
(C_3)	Compatibility	$xCy \text{ and } y \leq z \Rightarrow xCz$
(C_3)	Summation	$xC(y+z) \Rightarrow xCy \text{ or } xCz$

A Boolean algebra together with a contact relation defined on B, i.e., the structure $\langle B, C, +, \cdot, ^*, 0, 1\rangle$ is called a Boolean contact algebra (BCA).

Modal Logic

Definition

A modal logic frame (RL-frame) $\mathcal{F} = \langle W, R \rangle$ is a non-empty set W together with a binary relation R on W. A model $\mathcal{M} = \langle \mathcal{F}, v \rangle$ is a frame together with a valuation function $v : P \to \mathcal{P}(W)$.

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Definition

Let $\mathcal{M}$ be a model, $x \in W$, $\varphi \in Mod$. Then the satisfaction relation $\mathcal{M}, x \models \varphi$ is defined by

1. $\mathcal{M}, x \models [R]\varphi$ iff xRy implies $\mathcal{M}, y \models \varphi$ for all $y \in W$.

Relevance Logic

Definition

A relevance logic frame (RL-frame) $\mathcal{F} = \langle W, f, g \rangle$ is a non-empty set W together with a binary function f and a unary function g on W. A (relevance logic) model is a RL-frame together with a valuation function.

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Definition

Let $\mathcal{M}$ be a model, $x \in W$, $\varphi \in Mod$. Then the satisfaction relation $\mathcal{M}, x \models \varphi$ is defined by

1. $\mathcal{M}, x \models \varphi \twoheadrightarrow \psi$ iff $\mathcal{M}, y \models \varphi$ implies $\mathcal{M}, z \models \psi$ for all $y, z \in W$ with $x = f(y, z)$,
2. $\mathcal{M}, x \models \sim\varphi$ iff $\mathcal{M}, g(x) \not\models \varphi$.

Abbreviations

$$
\begin{array}{lcl}
\varphi \Wedge \psi & := & \neg(\varphi \twoheadrightarrow \neg\psi) \\
\varphi \Vee \psi & := & \neg\varphi \twoheadrightarrow \psi \\
N\varphi & := & {\sim}\neg\varphi \\
\varphi \multimap \psi & := & N(N\varphi \twoheadrightarrow N\psi) \\
\varphi \owedge \psi & := & \neg(\varphi \multimap \neg\psi) \\
\varphi \ovee \psi & := & \neg\varphi \multimap \psi \\
U & := & NE
\end{array}
$$

Relevance Logic with E

Definition

A relevance logic with E frame (PRLE-frame) $\mathcal{F} = \langle W, e, f, g \rangle$ so that $\langle W, f, g \rangle$ is a RL-frame and $e \in W$. A PRLE-model is a PRLE-frame together with a valuation function.

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Definition

Let $\mathcal{M}$ be a model, $x \in W$ be a state, and $\varphi \in PRLE$. Then the satisfaction relation $\mathcal{M}, x \models \varphi$ is defined by

1. $\mathcal{M}, x \models E$ iff $x = e$.

Characterization of Boolean Algebras

Theorem

A PRLE-frame $\mathcal{F}$ together with the definitions

$$\begin{array}{ll} x+y := f(x,y), & x \cdot y := f^d(x,y) = g(f(g(x),g(y))), \\ x^* := g(x), \quad 0 := e, & 1 := g(e) \end{array}$$

is a Boolean algebra iff the axiom schemas

1. $\varphi \rightarrow \sim\sim \varphi$,
2. $\varphi \curlywedge \psi \rightarrow \psi \curlywedge \varphi$,
3. $\varphi \rightarrow \varphi \curlywedge E$,
4. $\varphi \curlywedge (\psi \curlyvee \chi) \rightarrow (\varphi \curlywedge \psi) \curlyvee (\varphi \curlywedge \chi)$,
5. $\varphi \rightarrow \top \curlyvee (U \wedge (\varphi \curlywedge N\varphi))$

are true in $\mathcal{F}$.

Contact Axioms

Lemma

Let $\mathcal{F}$ be a BMRL-frame, i.e., a PRLE-frame in which the axiom from the previous lemma are valid. Then we have:

1. $\mathcal{F} \models [R]\neg E$ *iff* xRy *implies* $y \neq 0$ *for all* $x, y \in W$.
2. $\mathcal{F} \models \neg E \to ([R]\varphi \to \varphi)$ *iff* R *satisfies* C_1.
3. $\mathcal{F} \models \varphi \to [R]\langle R\rangle\varphi$ *iff* R *satisfies* C_2.
4. $\mathcal{F} \models [R]\varphi \to [R](\top \multimap \varphi)$ *iff* R *satisfies* C_3.
5. *If* R *satisfies* C_2, *then* $\mathcal{F} \models \varphi \to [R](\neg\langle R\rangle\varphi \twoheadrightarrow \langle R\rangle\varphi)$ *iff* R *satisfies* C_4.

Natural Deduction I

The natural deduction system works on three different kinds of formulas:

1. $\alpha = \beta$ where α, β are terms for Boolean algberas,
2. $\alpha C \beta$ where α, β are terms for Boolean algberas,
3. φ_α where φ is a PRLE formulas and α is a term for Boolean algberas.

Natural Deduction II

$$\frac{\alpha C \beta}{(\neg E)_\alpha}\ (BCA_{0l}) \qquad \frac{\alpha C \beta}{(\neg E)_\alpha}\ (BCA_{0r}) \qquad \frac{(\neg E)_\alpha}{\alpha C \alpha}\ (BCA_1) \qquad \frac{\beta C \alpha}{\alpha C \beta}\ (BCA_2)$$

$$\frac{\alpha C \beta \quad \beta = \beta \cdot \gamma}{\alpha C \gamma}\ (BCA_3) \qquad \frac{\alpha C(\beta + \gamma) \quad \begin{matrix}[\alpha C \beta] \\ \vdots \\ \chi\end{matrix} \quad \begin{matrix}[\alpha C \gamma] \\ \vdots \\ \chi\end{matrix}}{\chi}\ (BCA_4)$$

Natural Deduction III

$$\frac{\varphi_\beta \quad \psi_\gamma \quad \alpha = \beta + \gamma}{(\varphi \curlywedge \psi)_\alpha}\ (\curlywedge I) \qquad \frac{(\varphi \curlywedge \psi)_\alpha \qquad \begin{array}{c}[\varphi_x] \quad [\psi_y] \quad [\alpha = x + y] \\ \vdots \\ \chi\end{array}}{\chi}\ (\curlywedge E)^*$$

$$\frac{\begin{array}{c}[\alpha C y] \\ \vdots \\ \varphi_y\end{array}}{([C]\varphi)_\alpha}\ ([C]I)^{**} \qquad \frac{([C]\varphi)_\alpha \quad \alpha C \beta}{\varphi_\beta}\ ([C]E)$$

where the side conditions $*$ resp. $**$ require that x, y are new variables in the right subtree resp. that y is a new variable in the subtree.

Example

$$
\dfrac{
  \dfrac{
    [(\varphi \Wedge \psi)_z]^1 \qquad
    \dfrac{
      \dfrac{[(\varphi \twoheadrightarrow \neg\psi)_z]^2 \quad [\varphi_x]^3 \quad [z = x + y]^3}{(\neg\psi)_y}\ (\twoheadrightarrow E^5)
      \qquad [\psi_y]^3
    }{\bot}\ (\neg E^4)
  }{
    \dfrac{\bot}{\neg(\varphi \twoheadrightarrow \neg\psi)_z}\ (\neg I^2)
  }\ (\Wedge E^3)
}{((\varphi \Wedge \psi) \rightarrow \neg(\varphi \twoheadrightarrow \neg\psi))_z}\ (\rightarrow I^1)
$$

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3. The natural deduction system was implemented in Coq.

Thank you for your attention!

Questions?