Algebraic Solution of Weighted Minimax Single-Facility Constrained Location Problems

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Outline

Introduction: Tropical Mathematics and Optimization

- Tropical (idempotent) mathematics deals with the theory and applications of algebraic systems with idempotent operations

  A binary operation is idempotent, if, when applied to two equal values, it gives this value as the result (example: $\max(x, x) = x$)

- Tropical optimization is concerned with problems that are formulated and solved in the tropical mathematics setting

- The tropical optimization problems find applications in many areas to provide new solutions to various old and novel problems in

  - project scheduling, location analysis, transportation networks,
  - decision making, discrete event systems

Tropical Algebra: Max-Plus Algebra

Max-Plus Algebra

- Max-plus algebra $\mathbb{R}_{\max,+}$ is the set of reals with $-\infty$ adjoined, and two operations defined, called addition and multiplication
- Addition is denoted by $\oplus$ and given by

$$x \oplus y = \max(x, y)$$

- Addition has $-\infty$ as neutral element and the idempotent property

$$x \oplus x = \max(x, x) = x$$

- Multiplication is denoted by $\otimes$, has 0 as neutral element, and defined as

$$x \otimes y = x + y$$

Idempotent Semifield

- *Idempotent semifield:* the algebraic system $\langle \mathbb{X}, \oplus, \otimes, \mathbb{0}, \mathbb{1} \rangle$
- The carrier set $\mathbb{X}$ has neutral elements, *zero* $\mathbb{0}$ and *identity* $\mathbb{1}$
- The binary operations $\oplus$ and $\otimes$ are *associative and commutative*
- Multiplication $\otimes$ is *distributive* over addition
- Addition $\oplus$ is *idempotent:* $x \oplus x = x$ for all $x \in \mathbb{X}$
- Multiplication $\otimes$ is *invertible:* for each nonzero $x \in \mathbb{X}$, there exists an inverse $x^{-1} \in \mathbb{X}$ such that $x \otimes x^{-1} = \mathbb{1}$
  - In max-plus algebra $\mathbb{R}_{\max,+}$, the inverse x^{-1} corresponds to $-x$
- *Algebraic completeness:* the equation $x^r = a$ is solvable for any $a \in \mathbb{X}$ and real r (there exist powers with real exponents)
  - In max-plus algebra $\mathbb{R}_{\max,+}$, the power x^y corresponds to yx
- *Notational convention:* the multiplication sign $\otimes$ is omitted

Vectors over Idempotent Semifield

- *Vector addition* for two vectors $\boldsymbol{a} = (a_j)$ and $\boldsymbol{b} = (b_j)$ is given by

$$\{\boldsymbol{a} \oplus \boldsymbol{b}\}_j = a_j \oplus b_j \qquad (= \max(a_j, b_j))$$

- *Scalar multiplication* of a vector $\boldsymbol{a}$ by a scalar x is defined as

$$\{x\boldsymbol{a}\}_j = xa_j \qquad (= x + a_j)$$

- A vector without zero elements is *regular* (finite in $\mathbb{R}_{\max,+}$)
- *Multiplicative conjugate transposition* of a nonzero column vector $\boldsymbol{a} = (a_j)$ yields the row vector $\boldsymbol{a}^{-} = (a_j^{-})$, where

$$a_j^{-} = \begin{cases} a_j^{-1}, & \text{if } a_j \neq \mathbb{0}; \\ \mathbb{0}, & \text{otherwise} \end{cases} \qquad \left(= \begin{cases} -a_j, & \text{if } a_j \neq -\infty; \\ -\infty, & \text{otherwise} \end{cases} \right)$$

Matrices over Idempotent Semifield

- *Matrix addition* for two matrices $\boldsymbol{A} = (a_{ij})$ and $\boldsymbol{B} = (b_{ij})$ of the same size is given by

$$\{\boldsymbol{A} \oplus \boldsymbol{B}\}_{ij} = a_{ij} \oplus b_{ij} \qquad (= \max(a_{ij}, b_{ij}))$$

- *Matrix multiplication* for two matrices $\boldsymbol{A} = (a_{ij})$ and $\boldsymbol{C} = (c_{ij})$ of appropriate size is calculated as

$$\{\boldsymbol{A}\boldsymbol{C}\}_{ij} = \bigoplus_{k} a_{ik} c_{kj} \qquad \left(= \max_{k}(a_{ik} + c_{kj})\right)$$

- *Scalar multiplication* of a matrix $\boldsymbol{A}$ by a scalar x is given by

$$\{x\boldsymbol{A}\}_{ij} = x a_{ij} \qquad (= x + a_{ij})$$

- A matrix without zero columns is called *column-regular*

Square Matrices

- The *identity matrix* $\boldsymbol{I}$ has $\mathbb{1}$'s on the diagonal and $\mathbb{0}$'s elsewhere
- *Integer powers* routinely represent iterated matrix products as

$$\boldsymbol{A}^0 = \boldsymbol{I}, \qquad \boldsymbol{A}^p = \boldsymbol{A}\boldsymbol{A}^{p-1}, \qquad p \geq 1$$

- The *trace* of a square matrix $\boldsymbol{A} = (a_{ij})$ of order n is given by

$$\operatorname{tr} \boldsymbol{A} = a_{11} \oplus \cdots \oplus a_{nn}$$

- An idempotent analogue of *matrix determinant* is calculated as

$$\operatorname{Tr}(\boldsymbol{A}) = \operatorname{tr} \boldsymbol{A} \oplus \cdots \oplus \operatorname{tr} \boldsymbol{A}^n$$

- The *Kleene star* operator is given by

$$\boldsymbol{A}^* = \boldsymbol{I} \oplus \boldsymbol{A} \oplus \cdots \oplus \boldsymbol{A}^{n-1}, \qquad \operatorname{Tr}(\boldsymbol{A}) \leq \mathbb{1}$$

Vector Inequalities

- Given an $(m \times n)$-matrix $\boldsymbol{A}$ and m-vector $\boldsymbol{d}$, find n-vectors $\boldsymbol{x}$ to satisfy the inequality

$$\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{d}$$

Lemma

For any matrix $\boldsymbol{A}$ without zero columns, and regular vector $\boldsymbol{d}$, all solutions of the inequality are given by

$$\boldsymbol{x} \leq (\boldsymbol{d}^{-}\boldsymbol{A})^{-}$$

- Given an $(n \times n)$-matrix $\boldsymbol{A}$ and n-vector $\boldsymbol{b}$, find regular n-vectors $\boldsymbol{x}$ to satisfy the inequality

$$\boldsymbol{A}\boldsymbol{x} \oplus \boldsymbol{b} \leq \boldsymbol{x}$$

Theorem

For any matrix $\boldsymbol{A}$, the following statements hold:

(1) *if* $\mathrm{Tr}(\boldsymbol{A}) \leq \mathbb{1}$, *then all regular solutions of the inequality are given by*

$$\boldsymbol{x} = \boldsymbol{A}^{*}\boldsymbol{u}, \qquad \boldsymbol{u} \geq \boldsymbol{b};$$

(2) *if* $\mathrm{Tr}(\boldsymbol{A}) > \mathbb{1}$, *then there is no regular solution.*

Single-Facility Location Problems: Chebyshev Location

Constrained Minimax Weighted Chebyshev Location

- The *problem*: locate a new point in a feasible area to minimize the maximum weighted Chebyshev distance to given points
- The *Chebyshev distance* (maximum or l_∞ -metric) between two vectors $\boldsymbol{r} = (r_1, \ldots, r_n)^T$ and $\boldsymbol{s} = (s_1, \ldots, s_n)^T$ in $\mathbb{R}^n$ is given by

$$d(\boldsymbol{r}, \boldsymbol{s}) = \max_{1 \leq i \leq n} |r_i - s_i| = \max_{1 \leq i \leq n} \max\{r_i - s_i, s_i - r_i\}$$

- Suppose that *vectors* $\boldsymbol{r}_j = (r_{1j}, \ldots, r_{nj})^T \in \mathbb{R}^n$, positive real *weights* w_j, and real *addends* h_j are given for all $j = 1, \ldots, m$
- The *maximum weighted Chebyshev addended distance* from the unknown vector $\boldsymbol{x} = (x_1, \ldots, x_n)^T \in \mathbb{R}^n$ to the given vectors is

$$\max_{1 \leq j \leq m} (w_j d(\boldsymbol{x}, \boldsymbol{r}_j) + h_j) = \max_{1 \leq j \leq m} (w_j \max_{1 \leq i \leq n} |x_i - r_{ij}| + h_j)$$

Location Constraints

- Given m positive real *upper bounds* c_j, the constraints on the distance from the new point to the existing points are defined as

$$d(\boldsymbol{x},\boldsymbol{r}_j) = \max_{1\leq i\leq n}|x_i - r_{ij}| \leq c_j, \quad j = 1,\ldots,m$$

- These upper bound constraints specify, in the Cartesian system, a hyper-rectangle (if nonempty) with sides parallel to the axes
- Given a matrix $\boldsymbol{G} = (g_{ik}) \in \mathbb{R}^{n\times n}$ and vectors $\boldsymbol{p} = (p_i)$ and $\boldsymbol{q} = (q_i)$ in $\mathbb{R}^n$, the *feasible area* is defined by the constraints

$$g_{ik} + x_k \leq x_i, \qquad p_i \leq x_i \leq q_i, \qquad i,k = 1,\ldots,n$$

- This area is the intersection of half-spaces defined by the first inequalities and of the rectangle given by the double inequalities

Optimization Problem

- The location problem takes the form

$$
\begin{aligned}
\min_{x_1,\ldots,x_n} \quad & \max_{1\leq j\leq m}(w_j \max_{1\leq i\leq n}|x_i - r_{ij}| + h_j);\\
\text{s. t.} \quad & \max_{1\leq i\leq n}|x_i - r_{ij}| \leq c_j, \quad j = 1,\ldots,m;\\
& g_{ik} + x_k \leq x_i, \quad p_i \leq x_i \leq q_i, \quad i,k = 1,\ldots,n
\end{aligned}
$$

- The problem can be formulated and solved as a linear program by using, for instance, the simplex or Karmarkar algorithm
- This approach, however, provides a numerical solution, if it exists, rather than a direct, complete solution in an exact analytical form

Representation in Terms of Tropical Algebra

- In terms of $\mathbb{R}_{\max,+}$, the objective function is written as follows:

$$\max_{1\leq j\leq m}(w_j \max_{1\leq i\leq n}|x_i - r_{ij}| + h_j) \qquad \text{(in conventional algebra)}$$

$$\bigoplus_{1\leq j\leq m} h_j(\boldsymbol{x}^{-}\boldsymbol{r}_j \oplus \boldsymbol{r}_j^{-}\boldsymbol{x})^{w_j} \qquad \text{(in terms of } \mathbb{R}_{\max,+}\text{)}$$

- The upper bound constraints:

$$\max_{1\leq i\leq n}|x_i - r_{ij}| \leq c_j \qquad \text{(in conventional algebra)}$$

$$\boldsymbol{r}_j^{-}\boldsymbol{x} \oplus \boldsymbol{x}^{-}\boldsymbol{r}_j \leq c_j \qquad \text{(in terms of } \mathbb{R}_{\max,+}\text{)}$$

- The feasible area constraints:

$$g_{ik} + x_k \leq x_i, \quad p_i \leq x_i \leq q_i \qquad \text{(in conventional algebra)}$$

$$\boldsymbol{G}\boldsymbol{x} \leq \boldsymbol{x}, \quad \boldsymbol{p} \leq \boldsymbol{x} \leq \boldsymbol{q} \qquad \text{(in terms of } \mathbb{R}_{\max,+}\text{)}$$

Representation of Chebyshev Location Problem

- Conventional representation is as follows:

$$\begin{aligned} \min_{x_1,\ldots,x_n} \quad & \max_{1\leq j\leq m}(w_j \max_{1\leq i\leq n}|x_i - r_{ij}| + h_j); \\ \text{s. t.} \quad & \max_{1\leq i\leq n}|x_i - r_{ij}| \leq c_j, \quad j = 1,\ldots,m; \\ & g_{ik} + x_k \leq x_i, \quad p_i \leq x_i \leq q_i, \quad i,k = 1,\ldots,n \end{aligned}$$

- Tropical vector representation:

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & \bigoplus_{1\leq j\leq m} h_j(\boldsymbol{r}_j^{-}\boldsymbol{x} \oplus \boldsymbol{x}^{-}\boldsymbol{r}_j)^{w_j}; \\ \text{s. t.} \quad & \boldsymbol{r}_j^{-}\boldsymbol{x} \oplus \boldsymbol{x}^{-}\boldsymbol{r}_j \leq c_j, \quad j = 1,\ldots,m; \\ & \boldsymbol{G}\boldsymbol{x} \leq \boldsymbol{x}, \quad \boldsymbol{p} \leq \boldsymbol{x} \leq \boldsymbol{q} \end{aligned}$$

Theorem

Suppose that $\mathrm{Tr}(\boldsymbol{G}) \leq \mathbb{1}$ *and* $(c_i^{-1}\boldsymbol{r}_i^{-} \oplus \boldsymbol{q}^{-})\boldsymbol{G}^{*}(c_j^{-1}\boldsymbol{r}_j \oplus \boldsymbol{p}) \leq \mathbb{1}$ *for all* $i,j = 1,\ldots,m$*. Then, the minimum in the problem is equal to*

$$\theta = \bigoplus_{1\leq i,j\leq m} \Biggl((h_i^{1/w_i} h_j^{1/w_j} \boldsymbol{r}_i^{-}\boldsymbol{G}^{*}\boldsymbol{r}_j)^{\frac{w_i w_j}{w_i+w_j}} \oplus h_i(\boldsymbol{r}_i^{-}\boldsymbol{G}^{*}(c_j^{-1}\boldsymbol{r}_j \oplus \boldsymbol{p}))^{w_i}$$
$$\oplus h_j((c_i^{-1}\boldsymbol{r}_i^{-} \oplus \boldsymbol{q}^{-})\boldsymbol{G}^{*}\boldsymbol{r}_j)^{w_j} \Biggr),$$

and attained if and only if $\boldsymbol{x} = \boldsymbol{G}^{*}\boldsymbol{u}$*, where* $\boldsymbol{u}$ *satisfies the condition*

$$\bigoplus_{1\leq i\leq m} (\theta^{-1/w_i} h_i^{1/w_i} \oplus c_i^{-1})\boldsymbol{r}_i \oplus \boldsymbol{p} \leq \boldsymbol{u}$$
$$\leq \Biggl(\bigoplus_{1\leq i\leq m} ((\theta^{-1/w_i} h_i^{1/w_i} \oplus c_i^{-1})\boldsymbol{r}_i^{-} \oplus \boldsymbol{q}^{-})\boldsymbol{G}^{*} \Biggr)^{-}$$

Sketch of Proof

- Let θ denote the minimum value of the objective function
- All solutions of the problem are given by the system of inequalities

$$\bigoplus_{1\leq j\leq m} h_j(\boldsymbol{r}_j^{-}\boldsymbol{x}\oplus\boldsymbol{x}^{-}\boldsymbol{r}_j)^{w_j}\leq\theta,$$
$$\boldsymbol{r}_j^{-}\boldsymbol{x}\oplus\boldsymbol{x}^{-}\boldsymbol{r}_j\leq c_j,\quad j=1,\dots,m;$$
$$\boldsymbol{G}\boldsymbol{x}\leq\boldsymbol{x},\quad \boldsymbol{p}\leq\boldsymbol{x}\leq\boldsymbol{q}$$

- Solving the first inequalities with respect to $\boldsymbol{x}$ yields the system

$$\theta^{-1/w_j}h_j^{1/w_j}\boldsymbol{r}_j\leq\boldsymbol{x}\leq\theta^{1/w_j}h_j^{-1/w_j}\boldsymbol{r}_j,$$
$$c_j^{-1}\boldsymbol{r}_j\leq\boldsymbol{x}\leq c_j\boldsymbol{r}_j,\quad j=1,\dots,m;$$
$$\boldsymbol{G}\boldsymbol{x}\leq\boldsymbol{x},\qquad \boldsymbol{p}\leq\boldsymbol{x}\leq\boldsymbol{q}$$

- After combining the inequalities, the system becomes

$$\boldsymbol{G}\boldsymbol{x} \oplus \bigoplus_{1 \leq i \leq m} (\theta^{-1/w_i} h_i^{1/w_i} \oplus c_i^{-1}) \boldsymbol{r}_i \oplus \boldsymbol{p} \leq \boldsymbol{x}$$
$$\leq \left(\bigoplus_{1 \leq i \leq m} (\theta^{-1/w_i} h_i^{1/w_i} \oplus c_i^{-1}) \boldsymbol{r}_i^{-} \oplus \boldsymbol{q}^{-} \right)^{-}$$

- The left inequality is of the form $\boldsymbol{A}\boldsymbol{x} \oplus \boldsymbol{b} \leq \boldsymbol{x}$, and has the solution

$$\boldsymbol{x} = \boldsymbol{G}^{*} \boldsymbol{u}, \qquad \boldsymbol{u} \geq \bigoplus_{1 \leq i \leq m} (\theta^{-1/w_i} h_i^{1/w_i} \oplus c_i^{-1}) \boldsymbol{r}_i \oplus \boldsymbol{p}$$

- With the substitution $\boldsymbol{x} = \boldsymbol{G}^{*} \boldsymbol{u}$, the right inequality has the form $\boldsymbol{A}\boldsymbol{u} \leq \boldsymbol{d}$, and is solved as

$$\boldsymbol{u} \leq \left(\bigoplus_{1 \leq i \leq m} ((\theta^{-1/w_i} h_i^{1/w_i} \oplus c_i^{-1}) \boldsymbol{r}_i^{-} \oplus \boldsymbol{q}^{-}) \boldsymbol{G}^{*} \right)^{-}$$

- Combining the lower and upper bounds on the parameter vector $\boldsymbol{u}$ yields the solution in the form

$$\boldsymbol{x} = \boldsymbol{G}^{*}\boldsymbol{u}, \qquad \bigoplus_{1\leq i\leq m}(\theta^{-1/w_i}h_i^{1/w_i} \oplus c_i^{-1})\boldsymbol{r}_i \oplus \boldsymbol{p} \leq \boldsymbol{u}$$
$$\leq \left(\bigoplus_{1\leq i\leq m}((\theta^{-1/w_i}h_i^{1/w_i} \oplus c_i^{-1})\boldsymbol{r}_i^{-} \oplus \boldsymbol{q}^{-})\boldsymbol{G}^{*}\right)^{-}$$

- The parameter θ is derived by solving the inequality between the lower and upper bounds on the parameter vector $\boldsymbol{u}$, given by

$$\bigoplus_{1\leq i\leq m}(\theta^{-1/w_i}h_i^{1/w_i} \oplus c_i^{-1})\boldsymbol{r}_i \oplus \boldsymbol{p} \leq$$
$$\left(\bigoplus_{1\leq i\leq m}((\theta^{-1/w_i}h_i^{1/w_i} \oplus c_i^{-1})\boldsymbol{r}_i^{-} \oplus \boldsymbol{q}^{-})\boldsymbol{G}^{*}\right)^{-}$$

Rectilinear Location

Constrained Minimax Weighted Rectilinear Location

- The *problem*: locate a point in an area on the plane to minimize the maximum weighted rectilinear distance to given points
- The *rectilinear distance* (Manhattan, rectangular or l_1 -metric) between $\boldsymbol{r} = (r_1, r_2)^T$ and $\boldsymbol{s} = (s_1, s_2)^T$ in $\mathbb{R}^2$ is given by

$$d(\boldsymbol{r}, \boldsymbol{s}) = |r_1 - s_1| + |r_2 - s_2|$$

- Suppose that *vectors* $\boldsymbol{r}_j = (r_{1j}, r_{2j})^T \in \mathbb{R}^2$, positive real *weights* w_j and real *addends* h_j are given for all $j = 1, \ldots, m$
- The *maximum weighted rectilinear addended distance* from the unknown vector $\boldsymbol{x} = (x_1, x_2)^T \in \mathbb{R}^2$ to the given vectors is

$$\max_{1 \leq j \leq m} (w_j d(\boldsymbol{x}, \boldsymbol{r}_j) + h_j) = \max_{1 \leq j \leq m} (w_j(|x_1 - r_{1j}| + |x_2 - r_{2j}|) + h_j)$$

Location Constraints

- Given m positive real *upper bounds* c_j, the constraints on the distance from the new point to the existing points are defined as

$$d(\boldsymbol{x}, \boldsymbol{r}_j) = |x_1 - r_{1j}| + |x_2 - r_{2j}| \leq c_j, \quad j = 1, \dots, m$$

- These upper bound constraints specify in the plane a tilted rectangle (if nonempty) with sides at 45° to the axes
- Given real numbers p_1, p_2, q_1, q_2, a and b such that $p_1 \leq q_1$, $p_2 \leq q_2$, and $a \leq b$, the *feasible area* is defined by the constraints

$$p_1 - x_1 \leq x_2 \leq q_1 - x_1, \quad p_2 + x_2 \leq x_1 \leq q_2 + x_2, \quad a \leq x_2 \leq b$$

- This area is the intersection of the tilted rectangle given by the first two inequalities, and the horizontal strip given by the last inequality

Optimization Problem

- The rectilinear location problem in the standard form is given by

$$
\begin{aligned}
\min_{x_1,x_2} \quad & \max_{1\leq j\leq m} (w_j(|x_1 - r_{1j}| + |x_2 - r_{2j}|) + h_j); \\
\text{s. t.} \quad & |x_1 - r_{1j}| + |x_2 - r_{2j}| \leq c_j, \quad j = 1, \dots, m; \\
& p_1 - x_1 \leq x_2 \leq q_1 - x_1, \quad p_2 + x_2 \leq x_1 \leq q_2 + x_2, \\
& a \leq x_2 \leq b
\end{aligned}
$$

- In terms of $\mathbb{R}_{\max,+}$ (max-plus algebra), the problem becomes

$$
\begin{aligned}
\min_{x_1,x_2} \quad & \bigoplus_{1\leq j\leq m} h_j((r_{1j}^{-1}x_1 \oplus x_1^{-1}r_{1j})(r_{2j}^{-1}x_2 \oplus x_2^{-1}r_{2j}))^{w_j}; \\
\text{s. t.} \quad & (r_{1j}^{-1}x_1 \oplus x_1^{-1}r_{1j})(r_{2j}^{-1}x_2 \oplus x_2^{-1}r_{2j}) \leq c_j, \quad j = 1, \dots, m; \\
& p_1 x_1^{-1} \leq x_2 \leq q_1 x_1^{-1}, \quad p_2 x_2 \leq x_1 \leq q_2 x_2, \quad a \leq x_2 \leq b
\end{aligned}
$$

Vector Representation

- Vector-matrix notation is introduced as

$$\boldsymbol{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 x_2 \\ x_1 x_2^{-1} \end{pmatrix}, \quad \boldsymbol{p} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}, \quad \boldsymbol{q} = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix},$$

$$\boldsymbol{G} = \begin{pmatrix} \mathbb{0} & a^2 \\ b^{-2} & \mathbb{0} \end{pmatrix}, \quad \boldsymbol{s}_j = \begin{pmatrix} s_{1j} \\ s_{2j} \end{pmatrix} = \begin{pmatrix} r_{1j} r_{2j} \\ r_{1j} r_{2j}^{-1} \end{pmatrix}, \quad j = 1, \ldots, m$$

- With this notation, the rectilinear location problem is written as

$$\begin{aligned} \min_{y_1, y_2} \quad & \bigoplus_{1 \leq i \leq m} h_i (\boldsymbol{s}_i^- \boldsymbol{y} \oplus \boldsymbol{y}^- \boldsymbol{s}_i)^{w_i}; \\ \text{s. t.} \quad & \boldsymbol{s}_i^- \boldsymbol{y} \oplus \boldsymbol{y}^- \boldsymbol{s}_i \leq c_i, \quad i = 1, \ldots, m; \\ & \boldsymbol{G}\boldsymbol{y} \leq \boldsymbol{y}, \quad \boldsymbol{p} \leq \boldsymbol{y} \leq \boldsymbol{q} \end{aligned}$$

Theorem

Suppose that $(c_i^{-1}\boldsymbol{s}_i^{-} \oplus \boldsymbol{q}^{-})\boldsymbol{G}^{\ast}(c_j^{-1}\boldsymbol{s}_j \oplus \boldsymbol{p}) \leq \mathbb{1}$ *for all* $i, j = 1, \ldots, m$*. Then, the minimum value in the problem is equal to*

$$\theta = \bigoplus_{1 \leq i,j \leq m} \Big((h_i^{1/w_i} h_j^{1/w_j} \boldsymbol{s}_i^{-} \boldsymbol{G}^{\ast} \boldsymbol{s}_j)^{\frac{w_i w_j}{w_i + w_j}} \oplus h_i(\boldsymbol{s}_i^{-} \boldsymbol{G}^{\ast}(c_j^{-1}\boldsymbol{s}_j \oplus \boldsymbol{p}))^{w_i} \oplus h_j((c_i^{-1}\boldsymbol{s}_i^{-} \oplus \boldsymbol{q}^{-})\boldsymbol{G}^{\ast}\boldsymbol{s}_j)^{w_j} \Big),$$

and attained if and only if $x_1 = y_1^{1/2} y_2^{1/2}$ *and* $x_2 = y_1^{1/2} y_2^{-1/2}$*, where the vector* $\boldsymbol{y} = (y_1, y_2)^T$ *is given by* $\boldsymbol{y} = \boldsymbol{G}^{\ast}\boldsymbol{u}$ *under the condition*

$$\bigoplus_{1 \leq i \leq m} (\theta^{-1/w_i} h_i^{1/w_i} \oplus c_i^{-1})\boldsymbol{s}_i \oplus \boldsymbol{p} \leq \boldsymbol{u} \leq \Big(\bigoplus_{1 \leq i \leq m} ((\theta^{-1/w_i} h_i^{1/w_i} \oplus c_i^{-1})\boldsymbol{s}_i^{-} \oplus \boldsymbol{q}^{-})\boldsymbol{G}^{\ast} \Big)^{-}$$

Concluding Remarks

- The solution of the problems has a polynomial time complexity in the number of points m and the dimension of space n
- The most time consuming part is the calculation of the minimum θ
- The evaluation of θ requires calculating the matrix $\boldsymbol{G}^{*}$ with at most $O(n^4)$ operations if computed by direct matrix multiplications
- Under the condition that $m \geq n$, the overall number of arithmetic operations to compute θ is no more than $O(m^2 n^2)$
- The problems can be solved as linear programs using linear programming algorithms (eg, the simplex or Karmarkar algorithm)
- This approach, however, provides a numerical solution, if it exists, rather than a direct, complete solution in an exact analytical form
- The new direct solutions can be of interest when the application of algorithmic solutions appears to be impractical or impossible