Decidability for Subsignatures of Relation Algebra

Robin Hirsch

RAMICS, Groningen

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Signatures for operators on binary relations

Booleans

$\leq, \cdot, +, -$ (containment, intersection, union, complement)

Extra Operators

$1', \smile, ;$ (identity, converse, composition, maybe others)

Representation Classes

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- $F(S)$ is the closure under isomorphism of the class of S-structures of binary relations over a finite base, with concrete operations.

Decision Problems

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- Is $s = t$ valid over $R(S)$?
- Is $s = t$ valid over $F(S)$?

Known undecidable problems

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- Word problems

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- Tiling

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Known undecidable problems

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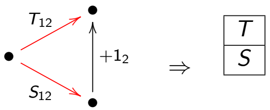
- Tiling
- Deterministic tiling
- Partial group embedding
- Partial group finite embedding.

Tiling reduction

Tiles $\tau \rightsquigarrow$ atoms $\{e_i, w_{ij}, c_{0k}, c_{k0}, +1_k, -1_k, T_{12}, T_{21} : i, j < 3,\ c = g, u, v,\ k = 1, 2,\ T \in \tau\}$
Forbidden triples; non-matching indices, (e_i, x, y) where $x \neq y$, and



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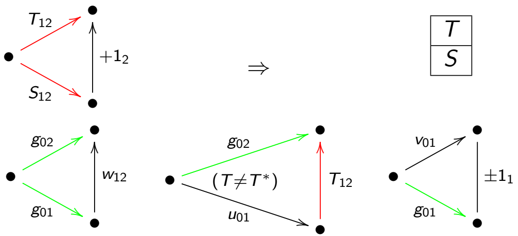

Partial group embedding problem

$*$	e	x	y
e	e	x	y
x	x	z	w
y	y	e	v

Partial group embedding problem

*	e	x	y	?
e	e	x	y	?
x	x	z	w	?
y	y	e	v	?
?	?	?	?	?

Partial group embedding problem

$*$	e	x	y	?	$\dots$
e	e	x	y	?	$\dots$
x	x	z	w	?	$\dots$
y	y	e	v	?	$\dots$
?	?	?	?	?	$\dots$
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$

Partial group reduction

$* : \sqrt{A} \times \sqrt{A} \to A$.

Atoms $\{e_{ii}, w_{ij} : i, j < 3\} \cup \{a_{01}, a_{12} : a \in \sqrt{A}\} \cup \{b_{02} : b \in A\}$.

Forbidden:

Partial group reduction

$* : \sqrt{A} \times \sqrt{A} \to A$.
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Forbidden:

- Indices don't match
- $(e_{ii}, x_{ij}, y_{ij}) : x \neq y$
- $(a_{01}, b_{12}, z_{02}) : z \neq a * b$
- $(a_{01}, w_{12}, (a * b)_{02})$

where $a, b \in \sqrt{A}$.

Membership of $R(S)$ for finite structures

	Operators	
Booleans	$; \notin S$	$; \in S$
$\emptyset$		
$\leq$		
$\cdot$		
$+$		
$\{\cdot, +\}^{\uparrow}$		

Membership decidable

Membership undecidable

Membership of $R(S)$ for finite structures

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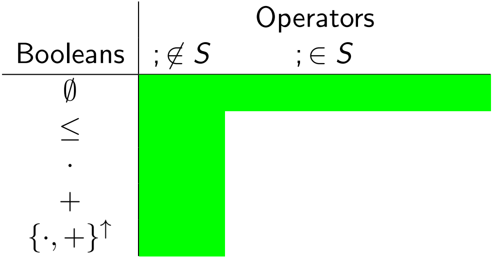

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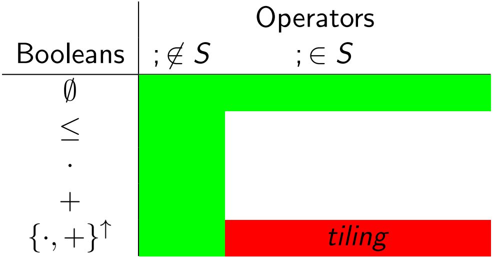

Membership decidable

Membership undecidable

Membership of $R(S)$ for finite structures

Booleans	Operators ; $\notin S$	; $\in S$
$\emptyset$		
$\leq$		
$\cdot$		$\smile \in S$ det. tiling
$+$		
$\{\cdot, +\}^{\uparrow}$		*tiling*

Membership decidable
Membership undecidable

Membership of $R(S)$ for finite structures

Booleans	Operators $; \notin S$	$; \in S$
$\emptyset$		
$\leq$		$- \in S$, $^\smile \notin S$ PG embed
$\cdot$		$^\smile \in S$ det. tiling
$+$		
$\{\cdot, +\}^\uparrow$		*tiling*

Membership decidabl

Membership undecida

Membership of $R(S)$ for finite structures

	Operators	
Booleans	$; \notin S$	$; \in S$
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$\leq$		$- \in S,\ ^{\smile} \notin S$ PG embed
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$+$		
$\{\cdot, +\}^{\uparrow}$		*tiling*

Membership decidabl
Membership undecida

Open: $\{+, ;\} \subseteq S,\ \cdot \notin S$

Membership of $F(S)$ for finite structures

	Operators		
Booleans	$; \notin S$	$(; \in S \& {}^{\smile} \notin S)$	$\{{}^{\smile}, ;\} \subseteq S$
$\emptyset$			
$\leq$			
$\cdot$			
$+$			
$\{\cdot, +\}^{\uparrow}$			

Membership of $F(S)$ for finite structures

	Operators		
Booleans	$; \notin S$	$(; \in S \,\&\, ^{\smile} \notin S)$	$\{^{\smile}, ;\} \subseteq S$
$\emptyset$			
$\leq$			
$\cdot$			
$+$			
$\{\cdot, +\}^{\uparrow}$			

Membership of $F(S)$ for finite structures

	Operators		
Booleans	$; \notin S$	$(; \in S \& ^\smile \notin S)$	$\{^\smile, ;\} \subseteq S$
$\emptyset$			
$\leq$			
$\cdot$			
$+$			
$\{\cdot, +\}^\uparrow$		Fin. Embedding	

Membership of $F(S)$ for finite structures

	Operators		
Booleans	$; \notin S$	$(; \in S \& ^\smile \notin S)$	$\{^\smile, ;\} \subseteq S$
$\emptyset$			
$\leq$			
$\cdot$		$- \in S$ Fin. Embedding	
$+$			
$\{\cdot, +\}^\uparrow$		Fin. Embedding	

Membership of $F(S)$ for finite structures

	Operators		
Booleans	$; \notin S$	$(; \in S \& ^{\smile} \notin S)$	$\{^{\smile}, ;\} \subseteq S$
$\emptyset$			
$\leq$			
$\cdot$		$- \in S$ Fin. Embedding	
$+$			
$\{\cdot, +\}^{\uparrow}$		Fin. Embedding	

Open: $S \supseteq \{^{\smile}, ;\}$ and $^{\smile} \notin S$, $; \in S$, $\emptyset \neq Bool(S) \subsetneq \{\leq, \cdot, ;\}$

Membership of $F(S)$ for finite structures

	Operators		
Booleans	$; \notin S$	$(; \in S \& ^{\smile} \notin S)$	$\{^{\smile}, ;\} \subseteq S$
$\emptyset$			
$\leq$		$- \in S$ Fin. Embedding	
$\cdot$			
$+$			
$\{\cdot, +\}^{\uparrow}$		Fin. Embedding	

Open: $S \supseteq \{^{\smile}, ;\}$ and $^{\smile} \notin S$, $; \in S$, $\emptyset \neq Bool(S) \subsetneq \{\leq, \cdot, ;\}$
Note: $S = \{\leq, D, R, ^{\smile}, ;\}$, $R(S), F(S)$ are fin. ax. [B77, HM13] and decidable.

Equational Theory of $R(S)$

	Operators	
Booleans	$; \in S$	$; \notin S$
$- \notin S$		
$- \in S \subsetneq BA$		
BA		

Equational Theory of $R(S)$

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Equational Theory of $R(S)$

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Equational Theory of $R(S)$

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Equational Theory of $F(S)$

	Operators	
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$- \notin S$		
$- \in S \subsetneq BA$		
BA		

Equational Theory of $F(S)$

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BA		

Equational Theory of $F(S)$

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BA		

Equational Theory of $F(S)$

	Operators	
Booleans	$; \in S$	$; \notin S$
$- \notin S$		
$- \in S \subsetneq BA$	$\smile \notin S$	
BA		

References

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